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AUTOMATION AND REMOTE CONTROL

Volume 19, Number 10 October, 1958

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TRANSIENT AND STEADY-STATE PROCESSES IN AN AUTOMATIC RANGE INDICATOR

PART I. DESCRIPTION OF THE DEVICE'S FUNCTIONING AND USE OF THE APPARATUS OF STEP FUNCTIONS IN THE ANALYSIS OF INDICATOR PROCESSES

F. M. Kilin

(Leningrad)

The transient and steady-state processes in an automatic range indicator are considered, account being taken of the specific idiosyncrasies characteristic of its functioning. Among these are the discontinuous character of the processes in the individual elements of the indicator, the variability of the design parameters which change by jumps in correspondence with the applied pulses, and the temporal transformations of pulses in the coincidence amplifiers.

1. Introduction

Automatic range indicators, conventionally called auto-range-finders (avtodal'nomer), are among the components of modern radar and radio-navigation systems. In these systems the auto-range-finder may execute various functions. We shall consider these functions in terms of an example, the auto-range-finder of a radar set.

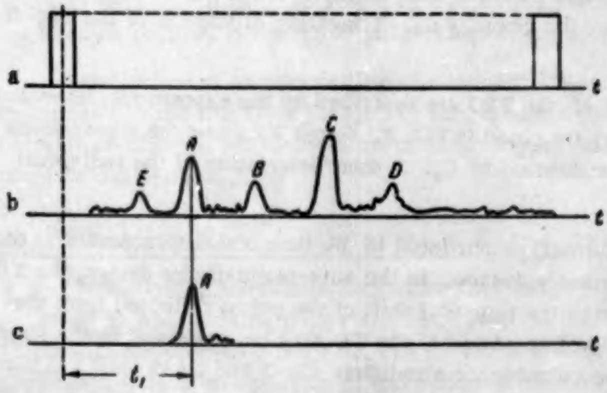

Fig. 1.

Figure 1,a shows what we shall call reference pulses, formed by circuits of the radar transmitter.

Figure 1,b shows pulses taken off the output of this radar's receiver. It follows from this display that there are some objects in the space scanned by this radar. In this case, an auto-range-finder can be used to solve the following problems.

1. Automatic tracking by the return pulses of any one object, for example, tracking on the A pulses (Fig. 1,b).

2. Automatic selection of signal pulses. If the auto-range-finder is tracking on the A pulses then, by means of this, it is possible to isolate the A pulses and to apply them to the circuits following the auto-range-finder (Fig. 1,c).

3. Automatic measurement of the time interval between the reference pulses and one of the return pulses on which the set is tracking. In the case of the auto-range-finder considered, the time interval measured would be that denoted by t_1 on Fig. 1.

2. Auto-Range-Finder Block Schematic

Figure 2 gives a simplified block schematic of a radar auto-range-finder. The auto-range-finder consists of two parts, a tracking part and a selection part. Included in the tracking part are a time discriminator TD, an operational amplifier OA, a regulated delay circuit RDC and a tracking pulse generator TPG.

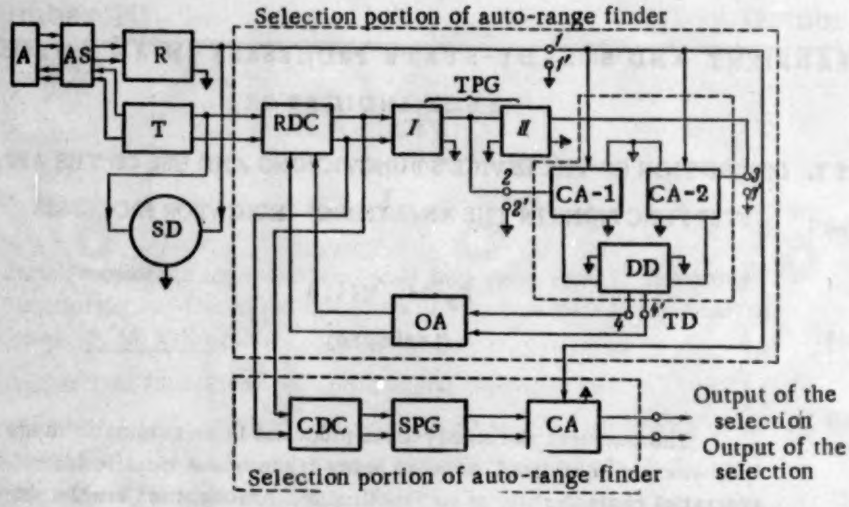

Fig. 2. Block schematic of an auto-range-finder device: A is the antenna, AS is the antenna switcher, R is the receiver, T is the transmitter and SD is the scope display.

Figure 3 provides graphs delineating the functioning of the auto-range-finder. Graphs 3,a represents the sequence of reference pulses with period T_0. These impulses are applied at the input of the RDC from the transmitter. The action of the reference pulses on the RDC causes the latter to form its own pulses, shown in Graph 3,b. These pulses are lagged in time by a certain amount with respect to the reference pulses. The magnitude of the pulse lag is proportional to the signal applied to the RDC from the operational amplifier. From the RDC, the lagged pulses are applied to the input of stage I of the TPG, and trigger it. As a result of this, stage I of the TPG forms pulses of height E_0 and length α_0, the leading edges of which coincide with the leading edges of the corresponding lagged pulses. In stage II of the TPG are formed pulses of the same height, EO, and length, α_0, as the pulses formed by stage I, but displaced from the latter by the quantity α_0. Triggering of stage II of the TPG is effected by the trailing edges of the stage I pulses.

The sequences of pulses formed by stages I and II of the TPG are described by the expressions, respectively, $E_I = E_0\ \gamma_1(t)$, $E_{II} = E_0\ \gamma_2(t)$. The graphs of $\gamma_1(t)$ and $\gamma_2(t)$ are given in Fig. 3. Graph 3,f shows the signal pulses from the receiver output which, in what follows, will be denoted by U_s. A short description of the individual elements in the auto-range-finder is given below.

The time discriminator generates a voltage (or current) proportional to the temporal displacement of the pulses of one sequence with respect to the pulses of another sequence. In the auto-range-finder design, the TD is a sensitive element by means of which may be detected the temporal shift of the pulses reflected from the target with respect to the tracking pulses. The principle of operation of the TD may be explained by the simplified circuit shown in Fig. 2. This circuit consists of two coincidence amplifiers Ca-1 and CA-2 and a differential detector. At the input terminals of the TD (1 and 1') are applied pulses $U_s(t)$ from the receiver, which we shall call return pulses. From the output terminals 4 and 4', is taken a voltage proportional to the temporal displacement of the return pulses with respect to the tracking pulses. The TD has additional terminals 2 and 2', 3 and 3', which receive pulses from the tracking pulse generator corresponding to E_I and E_{II}. The functioning of the TD consists of the following.

Each coincidence amplifier of the TD passes those portions of the target return pulses which coincide, in time, with the corresponding pulses applied to this coincidence amplifier from the TPG. Consequently, the cross-hatched

portion of pulse A, of length α_1 (Fig. 3,f), is applied to coincidence amplifier CA-1, and the remaining portion of this pulse, of length α_2, is applied to coincidence amplifier CA-2. Both coincidence amplifiers have identical circuits and parameters.

Figure 3,g shows the signal pulses, $U_s(t)$, after their temporal transformation in the coincidence amplifiers. The gain of the coincidence amplifiers is denoted by K_0. The quantity $\Delta\alpha[n]$ is the temporal shift of the n'th target pulse with respect to the center line of the tracking pulses with the same ordinal number. The pulse transformations are described by the expression $K_0U_s(t)\gamma(t)$, where $\gamma(t) = \gamma_1(t) - \gamma_2(t)$ (Fig. 3,e). Thus, in the passage of the return pulses from the target through the coincidence amplifiers, two sequences of pulses are formed and then applied to the differential detector DD (Fig. 2). In the differential detector, the difference of the constant components of these two pulse sequences is isolated. This difference is proportional to the magnitude of $\Delta\alpha[n]$.

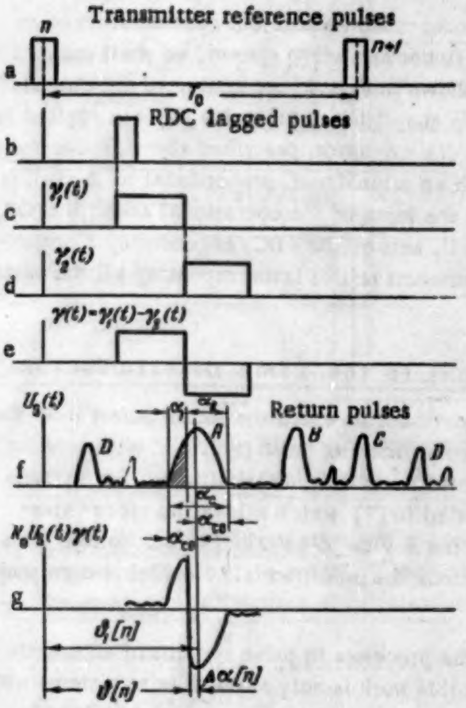

Fig. 3.

The operational amplifier amplifies the voltage formed in the circuits of the time discriminator, and transforms this voltage in accordance with some law described by the corresponding differential equation. The output voltage of the OA determines the magnitude of the pulse delay in the RDC.

The functions executed by the RDC and the TPG have been discussed above. Included in the selection portion of the auto-range-finder are a constant-delay circuit, CDC, a selection pulse generator, SPG, and a coincidence amplifier, CA (Fig. 2). The selection portion of the design is controlled by the delayed pulses of the CDC. Pulses are formed in the SPG which coincide in time with the pulses from that one of the targets which the auto-range-finder is supposed to track. The selector pulses, in turn, control the functioning of the coincidence amplifier, CA. As a result, of all the pulses applied to the CA from the receiver output, only those pulses are passed which are returns from the one target being tracked.

The temporal placement of a selector pulse with respect to the corresponding transmitter reference pulse depends on the magnitude of the signal at the OA output, i.e., it is continuously corrected by the tracking portion of the auto-range-finder. Therefore, the selection portion of the auto-range-finder, working in conjunction with the tracking portion, is conventionally called an automatic selector.

Descriptions of various designs of time discriminators, differential detectors and coincidence amplifiers are to be found in the literature [1-4]. For the regulated delay circuit in the auto-range-finder, wide use is made of phantastron circuits, phase-inverters of various types and other devices. Works [1, 3, 5] describe individual auto-range-finder designs.

3. The Auto-Range Finder as a Servo System

For analyzing the functioning of an auto-range-finder, it is convenient to estimate the temporal placement of the return and tracking pulses with respect to the corresponding reference pulses. Figure 3 showed the reference pulses with the ordinal number n. The center of a reference pulse is taken as the origin of coordinates. In this coordinate system, the position of the center of a return A pulse is determined by the magnitude of $\vartheta_1[n]$, which varies from one pulse to the next in correspondence with the variation in the ordinal number of the reference pulse.

On the same figure there is shown the position of the median line of the corresponding pair of tracking pulses, which is estimated by the magnitude of $\vartheta[n]$. If the auto-range-finder carried out ideal tracking, the median line of the tracking pulses would be exactly superposed on the median line of the return A pulses. Actually, the functioning of a real circuit is characterized by a tracking error, which is defined by the expression:

$$\Delta\alpha[n] = \vartheta_1[n] - \vartheta[n].$$

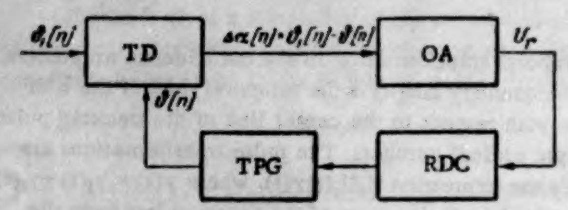

Fig. 4. Block schematic of an auto-range-finder as a servo system.

The quantities $\vartheta[n]$, $\vartheta_1[n]$ and $\Delta\alpha[n]$ are step functions. The independent variable $\underline{n}$ takes only integral values: $n = 0, 1, 2, \ldots$ A definition of step functions is given in [6]. In what follows we shall enclose the independent variable $\underline{n}$ of a step function in square brackets.

To render more graphic our consideration of an auto-range-finder as a servo system, we shall use as a base the circuit shown in Fig. 4. According to the circuit diagram, two quantities $\vartheta[n]$ and $\vartheta_1[n]$ are applied to the time discriminator (TD). In practice, the processes in the time discriminator, described above, reduce to a comparison of the magnitudes of $\vartheta[n]$ and $\vartheta_1[n]$, as a result of which an error signal, proportional to $\Delta\alpha[n]$, is formed at the output of the TD. The error signal is then applied to the input of the operational amplifier (OA) in which it is transformed to the quantity U_r. In its turn, the signal U_r acts on the RDC, engendering a proportional variation of the quantity ϑ. Such control of the quantity ϑ amounts to this latter repeating all the variations of the quantity ϑ_1.

4. Auto-Range-Finder Equations for Noiseless Input to the Time Discriminator

In this paper we consider a working regimen of the auto-range-finder in which the return pulses from the target being tracked coincide in time both with the first and the second tracking pulse (Fig. 3). With such an orientation of the return pulses, these latter are divided into two parts by the time discriminator. For investigating auto-range-finder dynamics, we use the methodology presented in [7] which allows the piece-wise-constant variability of the parameters in the individual elements of the device to be taken into account. This methodology is a further development of the apparatus of step functions, the fundamentals of which are presented in [6].

We mention in passing, that the apparatus for investigating the processes in pulse systems of automatic control is presented in [8]. However, the methodology described in this work is only applicable to systems with constant parameters. Moreover, the questions broached in this work are treated less completely than in [6].

We consider the functioning of an auto-range-finder in the case when the pulses reflected from the target are rectangular, or nearly so. With this assumption, the sequence of pulses applied to the input of the time discriminator can be described by the expressions:

$$U_s(t) = E_s\psi_0(t),$$

$$\psi_0(t) = \sum_{k=0}^{\infty} [H(t-kT) - H(t-\alpha-kT)].$$

Here, H(t) is a function defined by the equations:

$$H(t) = \begin{cases} 0 & \text{for } t \leqslant 1, \\ 1 & \text{for } t > 0, \end{cases}$$

$\psi_0(t)$ is a pulse function of unit height E_s, T and α are constant magnitudes of, respectively, the height, repetition period and length of the pulses reflected from the target (Fig. 5).

The tracking pulses, formed by stages I and II of the TPG, are described by pulse functions of the form:

$$\gamma_1(t) = \sum_{k=0}^{\infty} [H(t+\alpha_0-\Delta\vartheta_k-kT) - H(t-\Delta\vartheta_k-kT)],$$

$$\gamma_2(t) = \sum_{k=0}^{\infty} [H(t-\Delta\vartheta_k-kT) - H(t-\alpha_0-\Delta\vartheta_k-kT)],$$

$\Delta\vartheta_k$ is the magnitude of the variable pulse lag in the RDC. The pulse functions $U_s(t)$ and $\gamma(t) = \gamma_1(t) - \gamma_2(t)$, are shown on Fig. 5.

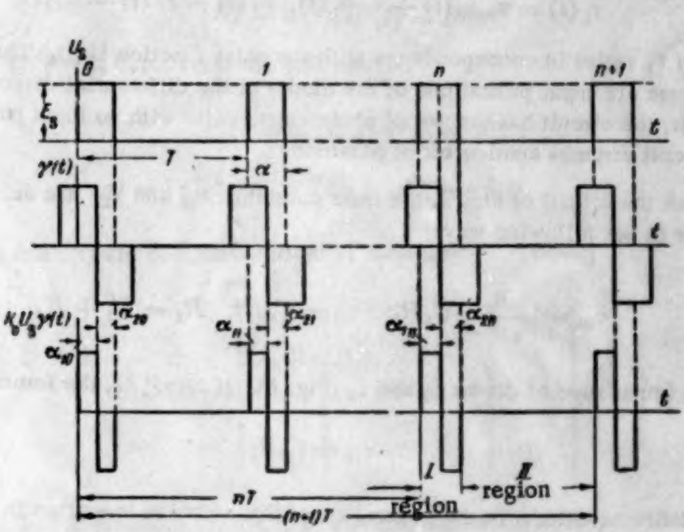

Fig. 5. Transformation of the signal pulses in the time discriminator.

Time discriminator equation. The coincidence amplifiers in the time discriminator implement the function of high-speed switching. In practice, their transient responses do not affect the magnitude of the output signals. Thus, the processes in the time discriminator reduce to the processes in the circuit shown in Fig. 6. In this circuit, switch K_1 carries out the function of coincidence amplifier CA-1, closing periodically for a length of time equal to the length of tracking pulses I. Switch K_2 carries out the function of coincidence amplifier CA-2, also closing periodically but during intervals of time equal in length to tracking pulses II. The voltages from the terminals of generators E_1 and E_2 are determined by the formula:

$$E_1 = -E_2 = K_0 E_s \psi_0 (t),$$

where K_0 is the gain of the coincidence amplifiers.

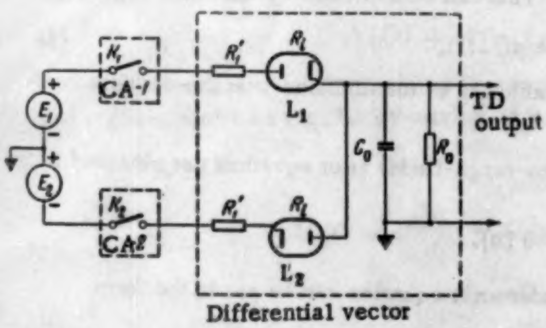

Fig. 6. Circuit equivalent ot the time discriminator.

Thus, both generators supply voltages which are equal in absolute value but opposite in polarity. The functioning of the time discriminator, in accordance with the circuit of Fig. 6, reduces to the following processes.

With switch K_1 closed, a portion of pulse U_s with the ordinal number n (Fig. 5), of length α_{1n} and coinciding with tracking pulses I, is passed by, and charges condenser C_0. The remaining portion of this pulse, of length α_{2n} and coinciding with tracking pulses II, is passed with negative polarity when switch K_2 is closed. With this condenser C_0 is discharged. As a result of the action of both pulses on condenser CO, a voltage is formed, proportional to the difference of the pulse lengths $\alpha_{1n} - \alpha_{2n}$. The lengths α_{1n} and α_{2n}, are related to the autorange-finder error in the following manner:

$$\alpha_{1n} = \frac{1}{2}\alpha + \Delta\alpha[n], \quad \alpha_{2n} = \frac{1}{2}\alpha - \Delta\alpha[n],$$
$$\alpha_{1n} - \alpha_{2n} = 2\Delta\alpha[n]. \tag{1}$$

Based on what has been presented, the equation of the time discriminator is written as:

$$[\tau_1(t)D + 1]\,U_1(t) = K_0 E_s \psi_0(t)\,\gamma(t). \tag{2}$$

Here, D = d/dt is the differentiation operator and τ_1 is a time constant which varies in accordance with the expression:

$$\tau_1(t) = \tau_{10}\psi_0(t) + \tau_{11}\psi_1(t), \quad \psi_1(t) = H(t) - \psi_0(t).$$

The magnitude of τ_1 varies in correspondence with the pulse function $U_s(t)$. This variation arises from the circumstance that, if there are input pulses, one of the diodes of the differential detector is conducting and, in correspondence with this, the circuit has one set of parameters, while with no input pulses, the diodes become non-conducting and the circuit assumes another set of parameters.

In accordance with the circuit of Fig. 6, the time constants τ_{10} and τ_{11} are defined by the parameters of the differential detector in the following way:

$$\tau_{10} = \frac{R_0}{R_0 + R_1} C_0 R_1, \quad \tau_{11} = C_0 R_0, \quad R_1 = R_1' + R_i,$$

where R_i is the internal impedance of diodes L_1 and L_2 (Fig. 6). If $R_0 \gg R_1$, the formula for τ_{10} assumes a simpler form:

$$\tau_{10} \approx C_0 R_1.$$

Operational amplifier equation. In the general case, this equation is written in the following form:

$$\sum_{k=1}^{r} [\delta_{jk} D + a_{jk}] U_k(t) = 0, \quad (j = 2, 3, \ldots, r) \tag{3}$$

where δ_{jk} is a Kronecker delta.

Equations of the regulated delay circuit (RDC). In the RDC, pulses are delayed by an amount proportional to the signal taken off the output of the operational amplifier. This can be expressed by the following equations:

$$\vartheta = \vartheta_0 + \Delta\vartheta, \quad \Delta\vartheta = \sigma U_r(t). \tag{4}$$

Here, ϑ_0 is the constant delay of the RDC, $\Delta\vartheta$ is the variable lag of the RDC and σ is the constant coefficient of proportionality.

Auto-range-finder error equation. In paragraph 3 the auto-range-finder error equation was obtained in the form:

$$\Delta\alpha[n] = \vartheta_1[n] - \vartheta[n].$$

With Equation (4) taken into account, the auto-range-finder error equation can be put in the form:

$$\Delta\alpha[n] + \Delta\vartheta[n] = \Delta\vartheta_1[n], \quad \Delta\vartheta_1[n] = \vartheta_1[n] - \vartheta_0. \tag{5}$$

5. Transformation of the Equations of the Time Discriminator and the Operational Amplifier

In accordance with (2) and (3), the processes in the time discriminator and the operational amplifier are described by the equations:

$$[D + a_{11}(t)] U_1(t) = f_1(t),$$

$$\sum_{k=1}^{r} [\delta_{jk} D + a_{jk}] U_k(t) = 0, \quad (j = 2, 3, \ldots, r). \tag{6}$$

Here,

$$a_{11}(t) = a_{11}^0 \psi_0(t) + a_{11}^1 \psi_1(t), \quad a_{11}^0 = \frac{1}{\tau_{10}}, \quad a_{11}^1 = \frac{1}{\tau_{11}},$$

$$f_1(t) = \frac{K_0 E_c}{\tau_{10}} \psi_0(t) \gamma(t).$$

To shorten the writing and the mathematical computations, we shall employ matrix notation. We write Equations (6) in matrix form

$$[DI + a(t)]\,U(t) = F(t). \tag{7}$$

Here, $I = ||\delta_{jk}||$ $(j, k = 1, 2, \ldots, r)$ is the unit matrix.

$$\begin{gathered} a(t) = \psi_0(t)\, a^0 + \psi_1(t)\, a^1, \\ a^0 = \| a^0_{jk} \|, \quad a^1 = \| a^1_{jk} \| \quad (j, k = 1, 2, \ldots, r). \end{gathered} \tag{8}$$

The functions U(t) and F(t) are column matrices of the form:

$$U(t) = \begin{Vmatrix} U_1(t) \\ U_2(t) \\ \cdots \\ \cdots \\ U_r(t) \end{Vmatrix}, \quad F(t) = \begin{Vmatrix} f_1(t) \\ 0 \\ \cdots \\ \cdots \\ 0 \end{Vmatrix}.$$

The symbols 0 and 1 in Expressions (8) are indicative of the fact that the coefficients correspond to the parameters of the system in the time intervals determined by the pulse functions $\psi_0(t)$ and $\psi_1(t)$. The matrix form of writing the equation in (7) is more general than the form used in (6), since the case is also included when the parameters of the operational amplifier have a piece-wise constant form of variability.

For the matrices, $DI + a^0$ and $DI + a^1$, we find the adjunct matrices, denoted by

$$A^0(D) = \| A^0_{jk}(D) \|, \quad A^1(D) = \| A^1_{jk}(D) \|,$$

and the determinants:

$$\Delta(DI + a^0) = |DI + a^0_{jk}|, \quad \Delta(DI + a^1) = |DI + a^1_{jk}|.$$

We now determine the function U(t) in the interval $nT < t \leq (n + 1)T$. This interval is decomposed into two regions, shown on Fig. 5. By solving Equation (7), we obtain the following expression for U(t):*

in region I

$$\begin{gathered} U(t) = e^{-a^0(t-nT)} U[n] + \frac{K_0 E_s}{\tau_{10}} [h(t - nT) - h(t - \alpha_{1n} - nT)], \\ h(t) = 0 \ \text{for} \ t < 0, \ nT < t \leqslant nT + \alpha; \end{gathered} \tag{9}$$

in region II

$$\begin{gathered} U(t) = e^{-a^1(t-\alpha-nT)} \left\{ e^{-a^0\alpha} U[n] + \frac{K_0 E_c}{\tau_{10}} [h(\alpha) - 2h(\alpha - \alpha_{1n})] \right\}, \\ nT + \alpha < t \leqslant (n + 1)T. \end{gathered} \tag{10}$$

Here,

$$U[n] = \{U(t)\}_{t=nT}. \tag{11}$$

The matrices $e^{-a^0 t}$ and $e^{-a^1 t}$ have the following expressions:

$$e^{-a^0 t} = \| \beta^0_{jk}(t) \|, \quad e^{-a^1 t} = \| \beta^1_{jk}(t) \|. \tag{12}$$

*The solutions of (9) and (10) may be obtained by the methods described in [9] in pp. 99-104.

In turn, the functions $\beta^0_{jk}(t)$ and $\beta^1_{jk}(t)$ are found from the operational representation:

$$L[\beta^s_{jk}(t)] = \frac{A^s_{jk}(p)}{\Delta(pI + a^s)}, \quad (s=0, 1).$$

In certain cases it is more convenient to express the matrices $e^{-a^0 t}$ and $e^{-a^1 t}$ in another form. In particular, they may be represented by series (Cf. [10], pp. 85-89):

$$\begin{aligned} e^{-a^0 t} &= I - \frac{a^0}{1!}t + \frac{(a^0)^2}{2!}t^2 - \ldots, \\ e^{-a^1 t} &= I - \frac{a^1}{1!}t + \frac{(a^1)^2}{2!}t^2 - \ldots \end{aligned} \tag{13}$$

The function h(t) is a column matrix of the form:

$$h(t) \doteq \begin{Vmatrix} h_1(t) \\ h_2(t) \\ \cdots \\ \cdots \\ h_r(t) \end{Vmatrix},$$

defined by the expressions:

$$h(t) = \int_0^t e^{-a^0(t-\tau)} f_0(\tau)\, d\tau, \quad f_0(t) = \begin{Vmatrix} H(t) \\ 0 \\ 0 \\ \cdots \\ \cdots \\ 0 \end{Vmatrix} \tag{14}$$

or, in expanded form, taking (12) into account:

$$h_j(t) = \int_0^t \beta^0_{j1}(t-\tau)\, d\tau.$$

The elements, $h_j(t)$, could also be found directly from the expressions

$$L[h_j(t)] = \frac{A^0_{j1}(p)}{F\Delta(pI + a^0)}.$$

where L is the operator of taking the Laplace transform.

By setting t = (n + 1)T in Equation (10), and introducing the nomenolature

$$b = e^{-a^1(T-\alpha)} e^{-a^0 \alpha},$$

$$\xi[n] = \frac{K_0 E_s}{\tau_{10}} e^{-a^1(T-\alpha)} [h(\alpha) - h(\alpha - \alpha_{1n})],$$

we obtain the recursion relationship

$$U[n+1] - bU[n] = \xi[n]. \tag{15}$$

from which the step function U[n] may be determined.

The step function U[n] is related to the function U(t) by Equation (11), i.e., it characterizes the variation of U(t) at the discrete moments of time, 0, T, 2T, ... In the intervals between these moments, the variation of U(t) is determined, for the respective regions, by Expressions (9) and (10), if we substitute in these equations the values of the step function U[n] which satisfy Equation (15). In the sequel, we shall call Equation (15) the discrete equation of the time discriminator of the operational amplifier.

The step function U[n] gives a rough representation of the character of the transient and steady-state processes in the system investigated. However, such a representation of the system's dynamic properties is completely sufficient for the solution of certain applied problems. For the investigation of processes in a system by means of the step function U[n], the initial Equation (7), with variable coefficients, is replaced by Equation (15), with constant coefficients, the solution and investigation of which pose no particular difficulty. The operational calculus methods for solving equations of the type of (15) are well developed, and are almost identical with the analogous method of solving differential equations with constant coefficients. Thanks to this, the analysis of pulse systems with piece-wise constant variability is simplified.

By means of the time discriminator and the operational amplifier, the retention and transformation into continuous signals proportional to $\Delta\alpha$ of the pulse signals are implemented. Such a type of transformation will be carried out in the given system if the matrices $e^{-a^0\alpha}$ and e^{-a^1T} are amenable to the approximate representations:

$$e^{-a^0\alpha} \approx I - a^0\alpha, \quad e^{-a^1T} \approx I - a^1T, \tag{16}$$

with this,

$$|a^0_{jk}\alpha| \ll 1, \quad |a^1_{jk}T| \ll 1. \tag{17}$$

Here, the signs $|\ |$ denote the absolute values of the products of the element a^0_{jk} by the pulse length α, and of the element a^1_{jk} by the repetition period T.

By substituting, in the expression for h(t) in (14), the matrix $e^{-a^0(t-\tau)}$ in the form of a series, as in (13), and by then carrying out the integration, we obtain:

$$h(t) = \left\{It - \frac{a^0}{2!}t^2 + \frac{(a^0)^2}{3!}t^3 - \dots\right\} f_0(t).$$

If Conditions (17) hold, we have

$$h(\alpha) \approx f_0\alpha, \quad f_0 = \begin{Vmatrix} 1 \\ 0 \\ 0 \\ \dots \\ \dots \\ 0 \end{Vmatrix}. \tag{18}$$

Equations (18) and (1) give us:

$$h(\alpha) - 2h(\alpha - \alpha_{1n}) \approx 2\Delta\alpha\,[n]\,f_0.$$

It follows from this that:

$$\xi\,[n] \approx \frac{2K_0E_3}{\tau_{10}}\,\Delta\alpha\,[n]\,f_0.$$

Thus, Equation (15) can be rewritten in the form:

$$U[n+1] - bU[n] = \frac{2K_0E_s}{\tau_{10}} \Delta\alpha[n] f_0. \tag{19}$$

According to (19), the form of the step function U[n] depends on the initial conditions for n = 0 and on the form of the step function $\Delta\alpha[n]$, which expresses the variation of the tracking error from one pulse to the next. Thanks to this, signals proportional to $\Delta\alpha[n]$ are formed in the elements of the given system.

6. Transient and Steady-State Processes in the Time Discriminator and Operational Amplifier

If the conditions in (17) hold, the expression for the coefficient b can be simplified, and given by the expression:

$$b = e^{-aT}, \tag{20}$$

where

$$a = \frac{a^0\alpha + a^1(T-\alpha)}{T}.$$

If we substitute the value of b in Equation (19), we get

$$U[n+1] - e^{-aT}U[n] = \frac{2K_0E_s}{\tau_{10}} \Delta\alpha[n] f_0. \tag{21}$$

If we denote by U[0] the initial condition for U[n] for n = 0, we find from Equation (21), the sequence U[1], U[2], . . . , U[n]. The expression for U[n] is written in the form:

$$U[n] = e^{-aTn}U[0] + \frac{2K_0E_s}{\tau_{10}} \sum_{k=0}^{n-1} e^{-aT(n-k-1)} \Delta\alpha[k] f_0. \tag{22}$$

The matrix e^{-at} has the expression

$$e^{-at} = \|\beta_{jk}(t)\| \quad (j,\ k = 1,\ 2,\ \ldots,\ r). \tag{23}$$

The functions $\beta_{jk}(t)$ are found by the operational transformations:

$$L[\beta_{jk}(t)] = \frac{A_{jk}(p)}{\Delta(qI + a)}, \tag{24}$$

where $A_{jk}(p)$ and $\Delta(pI + a)$ are, respectively, elements of the adjunct matrix and the determinant of the matrix pI + a. The elements a_{jk} are found from the formula:

$$a_{jk} = \frac{a_{jk}^0\alpha + a_{jk}^1(T-\alpha)}{T}.$$

We consider the general case, when the characteristic equation $\Delta(pI + a) = 0$ contains multiple roots. In this case,

$$\Delta(pI + a) = \prod_{\nu=1}^{N} (p + \lambda_\nu)^{q_\nu}, \qquad \sum_{\nu=1}^{N} q_\nu = r.$$

The operator functions in (24) can be represented in the form of sums of elementary fractions, i.e.,

$$\frac{A_{jk}(p)}{\Delta(pI+a)} = \sum_{\nu=1}^{N}\sum_{\rho=1}^{q_\nu} \frac{K_{\nu\rho}^{jk}}{(p+\lambda_\nu)^\rho}, \tag{25}$$

where

$$K_{\nu\rho}^{jk} = \frac{1}{(q_\nu-\rho)!}\left[\frac{d^{q_\nu-\rho}}{dp^{q_\nu-\rho}}\frac{(p+\lambda_\nu)^{q_\nu}A_{jk}(p)}{\Delta(pI+a)}\right]_{p=-\lambda_\nu}.$$

The adjunct matrix A(p) and its elements possess the following properties (Cf. [12], [13], pp. 74-80):

$$A_{jk}(-\lambda_\nu) = A_{jk}^{(1)}(-\lambda_\nu) = \ldots = A_{jk}^{(q_\nu-2)}(-\lambda_\nu) = 0,$$
$$A_{jk}^{(s)}(p) = \frac{d^s}{dp^s}A_{jk}(p). \tag{26}$$

On the basis of (26), we get

$$K_{\nu1}^{jk} = \frac{A_{jk}^{(q_\nu-1)}(-\lambda_\nu)}{(q_\nu-1)!}\left[\frac{(p+\lambda_\nu)^{q_\nu}}{\Delta(pI+a)}\right]_{p=-\lambda_\nu},$$
$$K_{\nu2}^{jk} = K_{\nu3}^{jk} = \ldots = K_{\nu q_\nu}^{jk} = 0. \tag{27}$$

By substituting (27) in (25), we get

$$\frac{A_{jk}(p)}{\Delta(pI+a)} = \sum_{\nu=1}^{N} K_{\nu1}^{jk}\frac{1}{p+\lambda_\nu}. \tag{28}$$

In Equation (28), we now make the transition from the transform to the original time domain, taking (23) and (24) into consideration. As a result, we obtain

$$e^{-at} = \|\beta_{jk}(t)\| = \|\sum_{\nu=1}^{N} K_{\nu1}^{jk}e^{-\lambda_\nu t}\|.$$

Consequently,

$$e^{-aTn} = \left\|\sum_{\nu=1}^{N} K_{\nu1}^{jk}e^{-\lambda_\nu Tn}\right\| \quad (j,\ k=1,\ 2,\ \ldots,\ r). \tag{29}$$

In order to put the matrix e^{-aTn} in the form of (29), we used the well-known operator transformation which underlies the transformation to canonical variables and normal coordinates [11].

By substituting the value of e^{-aTn} in Equation (22) and presenting it in expanded form, we get

$$U_j[n] = \sum_{\nu=1}^{N} e^{-\lambda_\nu Tn} \sum_{m=1}^{r} K_{\nu 1}^{jm} U_m[0] +$$

$$+ \frac{2K_0E_s}{\tau_{10}} \sum_{\nu=1}^{N} \sum_{k=0}^{n-1} K_{\nu 1}^{j1} e^{-\lambda_\nu T(n-k-1)} \Delta\alpha[k].$$

In the case when the step function $\Delta\alpha[n]$ has a constant value, i.e., $\Delta\ [n] = \Delta\alpha_0$, the expression for $U_j[n]$ takes the form:

$$U_j[n] = \sum_{\nu=1}^{N} e^{-\lambda_\nu Tn} \sum_{m=1}^{r} K_{\nu 1}^{jm} U_m[0] +$$

$$+ \frac{2K_0E_s\Delta\alpha_0}{\tau_{10}} \sum_{\nu=1}^{N} K_{\nu 1}^{j1} \frac{1-e^{-\lambda_\nu Tn}}{1-e^{-\lambda_\nu T}},$$

Expressions for $U_j[n]$ are easily found in other cases, if $\Delta\alpha[n]$ is given by elementary functions.

7. Discrete Equations of the Auto-Range-Finder

For an auto-range-finder considered as a closed system of automatic control, the discrete equations for the general case are written in the form:

equation of the time discriminator and operational amplifier

$$U_r[n] = \sum_{m=1}^{r} \beta_{rm}(nT) U_m[0] + \frac{2K_0E_s}{\tau_{10}} \sum_{k=0}^{n-1} \beta_{r1}(nT-kT-T) \Delta\alpha[k]; \tag{30}$$

equation of the regulated delay circuit

$$\Delta\vartheta[n] = \sigma U_r[n]; \tag{31}$$

equation for the auto-range-finder tracking error

$$\Delta\alpha[n] + \Delta\vartheta[n] = \Delta\vartheta_1[n]. \tag{32}$$

To obtain Equation (30), we employed Expressions (22), (20) and (23). Equations (31) and (32) were set up in correspondence with (4) and (5).

8. Operational Representation of the Auto-Range-Finder Discrete Equations

To solve the system of discrete equations obtained in paragraph 7, we use the method, presented in the book of Ia. Z. Tsypkin [6], of the operational calculus of step functions. We shall denote the discrete Laplace transform of a step function $f[n]$ in the following way:

$$L^*\{f[n]\} = \sum_{n=0}^{\infty} e^{-pn} f[n], \quad p = \varepsilon + i\omega, \quad \varepsilon > 0, \quad i = \sqrt{-1}.$$

We now apply the discrete Laplace transform to Equation (21). Using the prediction theorem (Cf. [6] pp. 21-23), we obtain

$$[e^p I - e^{-aT}] L^*\{U[n]\} = e^p U[0] + \frac{2K_0E_s}{\tau_{10}} f_0 L^*\{\Delta\alpha[n]\}. \tag{33}$$

Corresponding to (16) and (20), if Inequalities (17) hold, the matrix e^{-aT} can be given by the approximate expression $e^{-aT} \approx I - aT$. By substituting this expression in (33) we get

$$[PI + a] L^* \{U[n]\} = \frac{e^p}{T} U[0] + \frac{2K_0E_s}{\tau_{10}T} f_0 L^* \{\Delta\alpha[n]\}. \tag{34}$$

Here, $P = \dfrac{e^p - 1}{T}$.

We find from (34) that

$$L^* \{U_r[n]\} = \frac{e^p \sum_{m=1}^{r} A_{rm}(P) U_m[0]}{T\Delta(PI + a)} + \frac{2K_0E_s}{\tau_{10}T} \frac{A_{r1}(P)}{\Delta(PI + a)} L^* \{\Delta\alpha[n]\}. \tag{35}$$

Corresponding to (31) and (32), we also have:

$$\begin{gathered} L^* \{\Delta\vartheta[n]\} = \sigma L^* \{U_r[n]\}, \\ L^* \{\Delta\alpha[n]\} + L^* \{\Delta\vartheta[n]\} = L^* \{\Delta\vartheta_1[n]\}. \end{gathered} \tag{36}$$

By solving Equations (35) and (36) with respect to $L^* \{\Delta\alpha[n]\}$, we get

$$\begin{gathered} \left[\Delta a(P) + \frac{2\sigma K_0E_s}{\tau_{10}T} A_{r1}(P)\right] L^* \{\Delta\alpha[n]\} = \\ = \Delta a(P) L^* \{\Delta\vartheta_1[n]\} - \frac{\sigma e^p}{T} \sum_{m=1}^{r} A_{rm}(P) U_m[0], \\ \Delta a(P) = \Delta(PI + a). \end{gathered} \tag{37}$$

We now consider a continuous system with constant parameters, the processes in which are described by the equations:

$$\begin{gathered} [D + a_{11}] U_1(t) = \frac{2K_0E_s}{\tau_{10}T} \Delta\alpha(t), \\ \sum_{k=1}^{r} [\delta_{jk} D + a_{jk}] U_k(t) = 0, \qquad (j = 2, 3, \ldots, r), \\ \Delta\vartheta(t) = \sigma U_r(t); \quad \Delta\alpha(t) + \Delta\vartheta(t) = \Delta\vartheta_1(t). \end{gathered} \tag{38}$$

Applying the Laplace transform to Equations (38), and solving them for $L[\Delta\alpha(t)]$, we obtain

$$\begin{gathered} \left[\Delta a(p) + \frac{2\sigma K_0E_s}{\tau_{10}T} A_{r1}(p)\right] L[\Delta\alpha(t)] = \Delta a(p) L[\Delta\vartheta_1(t)] - \sigma \sum_{m=1}^{r} A_{rm}(p) U_k[0], \\ \Delta a(p) = \Delta(pI + a). \end{gathered} \tag{39}$$

By comparing Equations (37) and (39), we find a relationship subsisting between them, thanks to which we may simplify the task of finding the representation of the step function $\Delta\alpha[n]$.

9. Transition to the Discrete Equations from the Continuous Ones

We set

$$L^{*}\{\Delta\vartheta_1[n]\} = e^{p}\Theta_1(P), \quad \Delta a(P)\,\Theta_1(P) = \frac{Y(P)}{TP^{s}}. \tag{40}$$

In addition, we introduce the notation

$$X(P) = \Delta a(P) + \frac{2\sigma K_0 E_s}{\tau_{10}T} A_{r1}(P). \tag{41}$$

The functions X(P) and Y(P) are polynomials with constant coefficients:

$$X(P) = X_0P^{r} + X_1P^{r-1} + \ldots + X_r,$$
$$Y(P) = Y_0P^{s} + Y_1P^{s-1} + \ldots + Y_s.$$

We shall assume that the characteristic equation $X(P) = 0$ contains only simple roots. In this case, the function X(P) takes the form

$$X(P) = X_0\prod_{\nu=1}^{r}(P-\mu_\nu),$$

where the μ_ν are all different values.

Solving Equations (35) and (36) for $L^{*}\{\Delta\alpha[n]\}$, and taking (40) and (41) into account, we get

$$L^{*}\{\Delta\alpha[n]\} = \frac{e^{p}Y(P)}{TPX(P)} - \frac{\sigma e^{p}}{T}\sum_{m=1}^{r}\frac{A_{rm}(P)}{X(P)}U_m[0]. \tag{42}$$

We present the fraction in this equality in the form of a sum of elementary fractions:

$$\frac{Y(P)}{P^{s}X(P)} = \frac{G_0}{P^{s}} + \frac{G_1}{P} + \sum_{\nu=1}^{r}\frac{S_\nu}{P-\mu_\nu}. \tag{43}$$

The constant coefficients G_0, G_1 and S_ν are found by well-known formulae (Cf., for example, [13] chapter VI).

Taking (43) into account, we made the transition in Equation (42) from the transformed representation to the original.[1] As the result, we get

$$\Delta\alpha[n] = G_0Tn + G_1 + \sum_{\nu=1}^{r}S_\nu(1+\mu_\nu T)^{n} - \sigma\sum_{\nu=1}^{r}U'_\nu[0](1+\mu_\nu T)^{n}.$$

Here,

$$U'_\nu[0] = \sum_{m=1}^{r}\frac{A_{rm}(\mu_\nu)U_m[0]}{X'(\mu_\nu)},$$
$$X'(P) = \frac{d}{dP}X(P).$$

[1]Tables of inverse transforms, given on pages 211-217 of Ia. Z. Tsypkin's book [6], can be used in making the transition from the transformed equations to the original ones.

In the general case, the roots μ_ν can be either real or complex. If the following inequalities hold for them[1]

$$|\mathrm{Re}\,\mu_\nu T| \ll 1, \quad |\mathrm{Im}\,\mu_\nu T| \ll 1, \tag{44}$$

then $1 + \mu_\nu T \approx e^{\mu_\nu T}$.

Consequently,

$$\Delta\alpha[n] = G_0 Tn + G_1 + \sum_{\nu=1} S_\nu e^{\mu_\nu Tn} - \sigma \sum_{\nu=1}^{r} U'_\nu[0]\, e^{\mu_\nu Tn}. \tag{45}$$

In the case considered, the step function $\Delta\alpha[n]$ can be obtained from the continuous function $\Delta\alpha(t)$ in the following way:

$$\Delta\alpha[n] = [\Delta\alpha(t)]_{t=nT} \quad (n = 0, 1, 2, \ldots, \infty),$$

where

$$\Delta\alpha(t) = G_0 t + G_1 + \sum_{\nu=1} S_\nu e^{\mu_\nu t} - \sigma \sum_{\nu=1}^{r} U_\nu[0]\, e^{\mu_\nu t}.$$

The function $\Delta\alpha(t)$, in its turn, is found from the solution of a system of equations with constant coefficients of the form of (38), if $L[\vartheta_1(t)] = \Theta\Omega(p)$. In other words, if Conditions (44) hold, the investigation of processes in the auto-range-finder can be replaced by an investigation of the continuous processes in a system with constant parameters. This turns out to be valid, not only with respect to the variable $\Delta\alpha[n]$, but also for the other variables, $U_\nu[n]$ and $\Delta\vartheta[n]$. One can convince oneself of this by carrying out similar computations for the other variables also, computations which were omitted here due to their cumbersomeness.

Similar results can also be obtained for the case of multiple roots, if one uses the form of solving systems of linear differential equations obtained in the work of B. V. Bulgakov [11].

SUMMARY

The auto-range-finder equations, containing variable parameters varying step-wise, transform, in the general case, to a system of discrete equations with constant coefficients (30), (31) and (32), the solution and investigation of which pose no particular difficulty. Conditions were found for which the analysis of the auto-range-finder functioning reduces to the analysis of processes in some equivalent continuous system.

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ON SERVO SYSTEMS CONTAINING TWO PULSE ELEMENTS WITH UNEQUAL REPETITION PERIODS

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Pulse servo systems containing two pulse elements with unequal repetition periods are considered. Their equations and transfer functions are found. An investigation is carried out on the influence of the inequality in repetition periods of the two pulse elements on system stability. It is shown by an example that, in the general case, $T_1 = T_2$ (equal repetition periods of the two pulse elements) is not the most efficient regimen for the system from the point of view of increased stability.

Previously, systems were investigated which contained one or several pulse elements with identical repetition periods [3-5, 7-9] or with repetition periods which were simple multiples of one another. Systems containing several pulse elements with arbitrary repetition periods remained uninvestigated, despite the fact that this problem has essential theoretical and practical value.

1. Transfer Function of a System Containing Two Series-Connected Pulse Elements With Unequal Repetition Periods

Figure 1 shows an open-loop system with one pulse element. We denote by H(S) the transfer function of the linear portion, and by $h(t) = L^{-1}[H(S)]$ the impulsive response function of the system. If it is assumed that the porosity of the pulses $\gamma_0 \ll 1$, then the transfer function of the pulse system, with the condition that $h(0) = 0$, will have the form [6]:

$$H^*(ST_0, \varepsilon) = \frac{\gamma_0}{T_0} \sum_{l=-\infty}^{\infty} H(S + jl\omega_0)\, e^{jl\omega_0 T_0 \varepsilon},^* \tag{1}$$

where T_0 is the repetition period of the pulse element, $\omega_0 = 2\pi/T_0$ and the parameter ϵ varies within the interval $0 \le \epsilon < 1$.

It is clear from (1) that the transfer function of the system is

$$H^*(ST_0, 0) = \begin{cases} H(S) & \text{for} \quad T_0 = 0^{**}, \\ \dfrac{\gamma_0}{T_0} \sum\limits_{l=-\infty}^{\infty} H(S + jl\omega_0) & \text{for} \quad \varepsilon = 0 \text{ и } T \neq 0. \end{cases} \tag{7}$$

---

* If $h(0) \neq 0$ $\quad H^*(ST_0, \varepsilon) = \frac{\gamma_0}{T_0} \sum\limits_{l=-\infty}^{\infty} H(S + jl\omega_0)\, e^{jl\omega_0 T_0 \varepsilon} + \frac{h(0)}{2}$, or

$$H^*(ST_0, 0) = \frac{\gamma_0}{T_0} \sum_{l=-\infty}^{\infty} H(S + jl\omega_0) + \frac{h(0)}{2}.$$

**That is, without a pulse element.

where $H^*(ST_0, 0)$ is the discrete Laplace transform of the function $h(t) = L^{-1}[H(S)]$, i.e.,

$$H^*(ST_0, 0) = D\{h(t)\} = \sum_{m=0}^{\infty} h(mT_0)\, e^{-ST_0 m}.$$

We now consider the case of two pulse elements, connected in series, with corresponding linear portions (Fig. 2). The following relationships hold:

$$X(S) = C(S)\, H^*(ST_2, \varepsilon),\quad C(S) = E^*(ST_1, 0)\, G(S),$$

where

$$H^*(ST_2, \varepsilon) = \frac{\gamma_2}{T_2} \sum_{l=-\infty}^{\infty} H(S + jl\omega_2)\, e^{jl\omega_2 T_2 \varepsilon},$$

and T_1 and T_2 are the repetition periods of the first and second pulse elements.

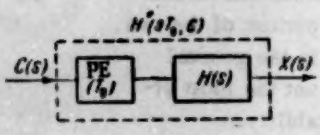

Fig. 1.

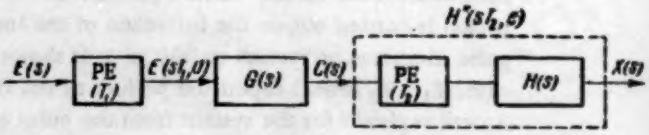

Fig. 2.

From the previous relationships, it follows that

$$X(S) = E^*(ST_1)\, G(S)\, H^*(ST_2, \varepsilon) =$$
$$= E^*(ST_1)\, G(S) \frac{\gamma_2}{T_2} \sum_{l=-\infty}^{\infty} H(S + jl\omega_2)\, e^{jl\omega_2 T_2 \varepsilon}. \tag{2}$$

If the discrete Laplace transform is applied to the function $x(t) = L^{-1}[X(S)]$ at the points mT_1, it is necessary to replace εT_2 by the values mT_1 in Equation (2).

Since $E^*(ST_1)$ is a periodic function, with period $\omega_1 = 2\pi/T_1$,

$$X^*(ST_1) = \frac{\gamma_1}{T_1} \sum_{m=-\infty}^{\infty} X(S + jm\omega_1) =$$
$$= E^*(ST_1) \frac{\gamma_1\gamma_2}{T_1 T_2} \sum_{m=-\infty}^{\infty} \sum_{l=-\infty}^{\infty} G(S + jm\omega_1)\, H(S + jl\omega_2 + jm\omega_1)\, e^{jlm\frac{\omega_2}{\omega_1}2\pi}.$$

The transfer function of the system in the open-looped state is given by the expression

$$\overline{K}W^*(S, T_1, T_2) = \frac{X^*(ST_1)}{E^*(ST_1)} =$$
$$= \frac{\gamma_1\gamma_2}{T_1 T_2} \sum_{m=-\infty}^{\infty} \sum_{l=-\infty}^{\infty} G(S + jm\omega_1)\, H(S + jl\omega_2 + jm\omega_1)\, e^{jlm\frac{\omega_2}{\omega_1}2\pi}. \tag{3}$$

For the system in the closed-loop state (Fig. 3), we have the equality:

$$E^*(ST_1) = F^*(ST_1) - X^*(ST_1). \tag{4}$$

The transfer functions of the closed-loop system, determined from (3) and (4), have the form:

$$\Phi^*(S, T_1, T_2) = \frac{X^*(ST_1)}{F^*(ST_1)} = \frac{\overline{K}W^*(S, T_1, T_2)}{1 + \overline{K}W^*(S, T_1, T_2)},$$
$$\Phi^*_\epsilon(S, T_1, T_2) = \frac{E^*(ST_1)}{F^*(ST_1)} = \frac{1}{1 + \overline{K}W^*(S, T_1, T_2)}. \tag{5}$$

We consider the following particular cases.

1. For $T_1 = T_2 = T_0$, Expression (2) takes the form:

$$\overline{K}W^*(ST_0) = \frac{\gamma_0}{T_0} \sum_{m=-\infty}^{\infty} G(S + jm\omega_0) \frac{\gamma_0}{T_0} \sum_{l=-\infty}^{\infty} H(S + jl\omega_0) = G^*(ST_0)\, H^*(ST_0).$$

2. For $T_1 = T_0$ and $T_2 = 0$ (i.e., without a second pulse element), it follows from (2) that

$$\overline{K}W^*(ST_0) = \frac{\gamma_0}{T_0} \sum_{m=-\infty}^{\infty} G(S + jm\omega_0)\, H(S + jm\omega_0).$$

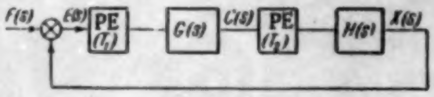

Fig. 3.

It is clear, from the argument presented above, that Formula (3) expresses the general transfer function of an open-looped pulse system. Since the linear portions, with transfer functions G(S) and H(S), are generally low-frequency filters then, for sufficiently large values of ω_1 and ω_2, it is possible to write

$$\sum_{m=-\infty}^{\infty} \sum_{l=-\infty}^{\infty} G(S + jm\omega_1)\, H(S + jl\omega_2 + jm\omega_1)\, e^{\left(jlm \frac{\omega_2}{\omega_1} 2\pi\right)} \approx$$
$$\approx G(S)\, H(S) + G(S + j\omega_1)\Big[H(S + j\omega_1) + H[S + j(\omega_1 - \omega_2)]\, e^{-j\frac{\omega_2}{\omega_1}2\pi} +$$
$$+ H[S + j(\omega_1 - 2\omega_2)]\, e^{-j2\left[\frac{\omega_2}{\omega_1}2\pi\right]}\Big] + G(S - j\omega_1)\Big[H(S - j\omega_1) + \tag{6}$$
$$+ H[S - j(\omega_1 - \omega_2)]\, e^{-j\frac{\omega_2}{\omega_1}2\pi} + H[S - j(\omega_1 - 2\omega_2)]\, e^{-j2\frac{\omega_2}{\omega_1}2\pi}\Big].$$

This formula permits the frequency characteristic to be constructed approximately, after which the stability and quality of the pulse system can be investigated for arbitrary values of T_1 and T_2.

2. Equation of a Discrete Correcting Device Consisting of Two Pulse Elements. The Repetition Period of the Second is a Divisor of the Repetition Period of the First

We assume that the repetition period of the second pulse element T_2 is an integral divisor of the first, i.e., $T_1 = T_0$ and $T_2 = T_1/n$, where n is an integer. In this case, (3) takes the following form:

$$\overline{K}W^*(S, T_1, T_2) = \frac{\gamma_1\gamma_2}{T_1T_2} \sum_{m=-\infty}^{\infty} \sum_{l=-\infty}^{\infty} G(S + jm\omega_1)\, H\left[S + j\omega_2\left(l + \frac{m}{n}\right)\right]. \tag{7}$$

In this case, the linear portions of the pulse system, with transfer functions G(S) and H(S), can be interchanged without changing the over-all system transfer function. To convince ourselves of this, we prove the

validity of the following identity:

$$\frac{\gamma_1\gamma_2}{T_1T_2}\sum_{m=-\infty}^{\infty}\sum_{l=-\infty}^{\infty} G(S+jm\omega_1)\,H\left[S+j\omega_2\left(l+\frac{m}{n}\right)\right]=$$

$$=\frac{\gamma_1\gamma_2}{T_1T_2}\sum_{m=-\infty}^{\infty}\sum_{l=-\infty}^{\infty} H(S+jm\omega_1)\,G\left[S+j\omega_2\left(l+\frac{m}{n}\right)\right]. \tag{8}$$

To prove this last identity, we assume that $(l + m/n)\omega_2 = M\omega_1$ and $m\omega_1 = (L + M/n)\omega_2$. It follows from this that $l = -L$ and $m = nL + M$.

By substituting the values of l and $\underline{m}$ just obtained in (7), we find that

$$\overline{KW}^*(S,T_1,T_2)=\frac{\gamma_1\gamma_2}{T_1T_2}\sum_{nL+M=-\infty}^{\infty}\sum_{L=-\infty}^{\infty} G\left[S+j\left(L+\frac{M}{n}\right)\omega_2\right]H(S+jM\omega_1)=$$

$$=\frac{\gamma_1\gamma_2}{T_1T_2}\sum_{m=-\infty}^{\infty}\sum_{l=-\infty}^{\infty} G\left[S+j\left(l+\frac{m}{n}\right)\omega_2\right]H(S+jm\omega_1).$$

From other considerations, we obtain (8).

It follows from Fig. 2 that

$$C(S)=E^*(ST_1)\,G(S),\quad X(S)=C^*(ST_2)\,H(S). \tag{9}$$

It is known that

$$C^*(ST_2)=\frac{\gamma_2}{T_2}\sum_{l=-\infty}^{\infty} C(S+jl\omega_2)=$$

$$=\frac{\gamma_2}{T_2}\sum_{l=-\infty}^{\infty} E^*(ST_1+jl\omega_2)\,G(S+jl\omega_2)=$$

$$=\frac{\gamma_1}{T_1}\frac{\gamma_2}{T_2}\sum_{m=-\infty}^{\infty}\sum_{l=-\infty}^{\infty} E(S+jm\omega_1+jl\omega_2)\,G(S+jl\omega_2)=$$

$$=\frac{\gamma_1}{T_1}\frac{\gamma_2}{T_2}\sum_{m=-\infty}^{\infty}\sum_{l=-\infty}^{\infty} E[S+j\omega_1(m+nl)]\,G(S+jl\omega_2)=$$

$$=\frac{\gamma_1}{T_1}\frac{\gamma_2}{T_2}\sum_{m=-\infty}^{\infty}\sum_{l=-\infty}^{\infty} E(S+jm\omega_1)\,G(S+jl\omega_2)=$$

$$=E^*(ST_1)\,G^*(ST_2).$$

By substituting the expression for $G^*(ST_2)$ in (9), we find:

$$X(S)=E^*(ST_1)\,G^*(ST_2)\,H(S),$$

$$X^*(ST_1)=\frac{\gamma_1}{T_1}\sum_{m=-\infty}^{\infty} X(S+jm\omega_1)=$$

$$=E^*(ST_1)\frac{\gamma_1\gamma_2}{T_1T_2}\sum_{m=-\infty}^{\infty}\sum_{l=-\infty}^{\infty} G\left[S+j\omega_2\left(l+\frac{m}{n}\right)\right]H(S+jm\omega_1).$$

Expression (7) can be rewritten in the form

$$X^*(ST_1) = E^*(ST_1)\,\frac{\gamma_1\gamma_2}{T_1T_2}\sum_{m=-\infty}^{\infty}\sum_{l=-\infty}^{\infty} G(S + jm\omega_1)\, H\left[S + j\omega_2\left(l + \frac{m}{n}\right)\right].$$

By combining the last two expressions for $X^*(ST_1)$, we obtain Identity (8), q.e.d.

We now assume that $\mu(t) = L^{-1}[G(S)H^*(ST_0)/n)]$ and $\eta(t) = L^{-1}[H(S)G^*(ST_0/n)]$. Identity (8) shows that the discrete Laplace transforms of these two functions are equal, i.e.,

$$\overline{K}W^*(ST_0) = D\{\mu(t)\} = D\{\eta(t)\},$$

or

$$\overline{K}W^*(ST_0) = D\left\{L^{-1}\left[G(S)\,H^*\left(S\frac{T_0}{n}\right)\right]\right\} = D\left\{L^{-1}\left[H(S)\,G^*\left(S\frac{T_0}{n}\right)\right]\right\}.$$

The last two expressions for $\overline{K}^*(ST_0)$ corresponds to Fig. 4,a and Fig. 4,b. If, in these diagrams, we introduce a third pulse element with period T_0/n before the element with period T_0 (Fig. 4,c), the systems considered do not change their properties and, consequently, the last expression can be rewritten in the following form:

$$\overline{K}W^*(ST_0) = D\{D_n^{-1}[G^*(ST_0/n)\,H^*(ST_0/n)]\},$$

where we denote by D_n^{-1} the operator for the inverse discrete Laplace transform, and $G^*(ST_0/n)$ and $H^*(ST_0/n)$ are the discrete Laplace transforms of the functions $g(t) = L^{-1}[G(S)]$ and $h(t) = L^{-1}[H(S)]$ with period T_0/n, i.e.,

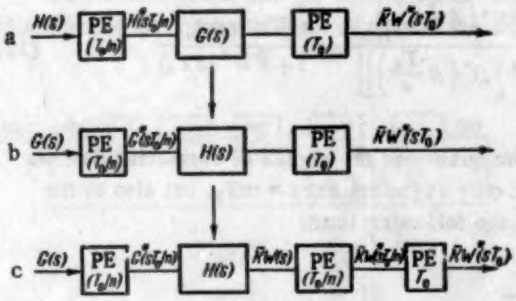

Fig. 4.

$$G^*\left(S\frac{T_0}{n}\right) = D_n\{g(t)\} = \sum_{m=0}^{\infty} g\left(\frac{mT_0}{n}\right)e^{-\frac{ST_0m}{n}},$$

$$H^*\left(S\frac{T_0}{n}\right) = D_n\{h(t)\} = \sum_{m=0}^{\infty} h\left(\frac{mT_0}{n}\right)e^{-\frac{ST_0m}{n}}.$$

If the transfer functions of the linear portions are known,

$$C(S) = \sum_j \frac{c_j}{S + a_j}, \quad H(S) = \sum_i \frac{d_i}{S + b_i}e^{-\tau S},$$

then $\overline{K}W^*(ST_0)$ can be computed from the formula (Cf. Appendix):

$$\overline{K}W^*(ST_0) = \gamma_1\gamma_2\sum_j\sum_i \frac{c_jd_ie^{-b_i\Delta_1\frac{T_0}{n}}e^{-ST_0m_2}}{e^{-a_j\frac{T_0}{n}} - e^{-b_i\frac{T_0}{n}}}\left[\frac{e^{ST_0}e^{-a_j\Delta_2T_0}}{e^{ST_0} - e^{-a_jT_0}} - \frac{e^{ST_0}e^{-b_i\Delta_2T_0}}{e^{ST_0} - e^{-b_iT_0}}\right],$$

where τ is a constant value of lag, and Δ_1, Δ_2 and m_2 are as shown on Fig. 8.

In the case when some poles of G(S) and H(S) coincide, i.e., for

$$G(S) = \sum_\lambda \frac{c_\lambda}{S + a_\lambda} + \sum_j \frac{c_j}{S + a_j},$$

$$H(S)=\sum_{\lambda}\frac{d_\lambda}{S+a_\lambda}e^{-\tau S}+\sum_{i}\frac{d_i}{S+b_i}e^{-\tau S},$$

$\overline{K}W^*(ST_0)$ can be given as the sum of two terms $\overline{K}W(ST_0) = \overline{K}W_1^*(ST_0) + \overline{K}W_2^*(ST_0)$ the second of which, $\overline{K}W_2^*(ST_0)$, (the sum of products of terms with nonidentical poles), is determined from Formula (10) and the first of which $\overline{K}W_1^*(ST_0)$ (the sum of products of terms with identical poles), is determined from the following formula (Cf. Appendix):

$$\overline{K}W_1^*(ST_0)=$$
$$=\gamma_1\gamma_2 n\sum_{\lambda}\frac{c_\lambda d_\lambda e^{-a_\lambda\Delta_1\frac{T_0}{n}}e^{(1-m_0)ST_0}e^{-a_\lambda\Delta_2 T_0}\left[e^{-a_\lambda T_0}+\Delta_2\left(e^{ST_0}-e^{-a_\lambda T_0}\right)\right]}{e^{-a_\lambda\frac{T_0}{n}}\left(e^{ST_0}-e^{-a_\lambda T_0}\right)^2}. \tag{11}$$

In this case, the transfer function of the closed-loop system has the form:

$$\Phi^*(ST_0)=\frac{X^*(ST_0)}{F^*(ST_0)}=\frac{D\left\{D_n^{-1}\left[G^*\left(S\frac{T_0}{n}\right)H^*\left(S\frac{T_0}{n}\right)\right]\right\}}{1+D\left\{D_n^{-1}\left[G^*\left(S\frac{T_0}{n}\right)H^*\left(S\frac{T_0}{n}\right)\right]\right\}}=\frac{\overline{K}W^*(ST_0)}{1+\overline{K}W^*(ST_0)} \tag{12}$$

It is possible, using (10)-(12), to carry out an analysis of the quality of the choice of correcting devices for such systems. If we are interested in the output quantity, not only at the instant $t = mT_0$, but also at the instant $t = mT_0/n$ (m and n integers), then (12) can be written in the following form:

$$X^*\left(S\frac{T_0}{n}\right)=\frac{G^*\left(S\frac{T_0}{n}\right)H^*\left(S\frac{T_0}{n}\right)}{1+D\left\{D_n^{-1}\left[G^*\left(S\frac{T_0}{n}\right)H^*\left(S\frac{T_0}{n}\right)\right]\right\}}F^*(ST_0)=$$
$$=\frac{\overline{K}W^*\left(S\frac{T_0}{n}\right)}{1+D\left\{D_n^{-1}\left[G^*\left(S\frac{T_0}{n}\right)H^*\left(S\frac{T_0}{n}\right)\right]\right\}}F^*(ST_0). \tag{13}$$

This expression coincides with the result given in work [1].

Using the notation $X^*(ST_0/n)/F^*(ST_0) = \Phi_1^*(ST_0/n)$, we write (13) in the form

$$\Phi_1^*\left(S\frac{T_0}{n}\right)=\frac{\overline{K}W^*\left(S\frac{T_0}{n}\right)}{1+D\left\{D_n^{-1}\left[KW^*\left(S\frac{T_0}{n}\right)\right]\right\}},$$

or, alternatively,

$$D\left\{D_n^{-1}\left[\overline{K}W^*\left(S\frac{T_0}{n}\right)\right]\right\}\Phi_1^*\left(S\frac{T_0}{n}\right)-\overline{K}W^*\left(S\frac{T_0}{n}\right)=-\Phi_1^*\left(S\frac{T_0}{n}\right),$$

where $\overline{K}W^*(ST_0/n)$ is the desired function.

The solution of this equation (as shown in [1]) has the form:

$$\overline{K}W^{*}\left(S\frac{T_0}{n}\right)=\frac{\Phi_1^{*}\left(S\frac{T_0}{n}\right)}{1-\Phi^{*}(ST_0)}.$$

Whence,

$$G^{*}\left(S\frac{T_0}{n}\right)=\frac{1}{H^{*}\left(S\frac{T_0}{n}\right)}\frac{\Phi_1^{*}\left(S\frac{T_0}{n}\right)}{1-\Phi_1^{*}(ST_0)}. \tag{14}$$

The relation (14) determines the transfer function correction for the transfer function object H(S) and functions $\Phi_1^{*}\ S\frac{T_0}{n}$ and $\Phi^{*}(ST_0)$.

3. Investigation of the Influence on System Stability of the Inequality of the Repetition Periods of the Pulse Elements

As an example of the use of a discrete correcting device of two pulse elements (the repetition period of the second being an integral divisor of that of the first), we consider the system whose block schematic is given in Fig. 5. We assume that the system possesses the following properties: 1) it has first-order astatism and 2) it provides the minimum possible regulation time. It is required to determine the form of the correcting device.

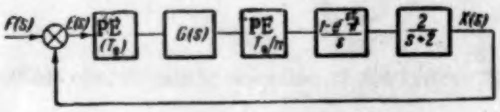

Fig. 5.

We consider the following cases.

a) $n = 2$, $T_0 = 1$ second. We determine $H^{*}(ST_0/2)$ in the ordinary way:

$$H^{*}\left(S\frac{T_0}{2}\right)=D_2\left\{L^{-1}\left[\frac{2\left(1-e^{-\frac{ST_0}{2}}\right)}{S(S+2)}\right]\right\}=\frac{0{,}63}{e^{\frac{ST_0}{2}}-0{,}37}. \tag{15}$$

It is known [1] that for a system with astatism of the γ'th order, the transfer functions $\Phi^{*}(ST_0)$ and $\Phi_1^{*}(ST_0/n)$ have the following forms [1]:

$$1-\Phi^{*}(ST_0)=\frac{(1-e^{-ST_0})^{\gamma}A^{*}(ST_0)}{1+\sum\limits_{k=1}^{n}c_k e^{-ST_0k}},$$

$$\Phi_1^{*}(ST_0/n)=\frac{\left[1+e^{-\frac{ST_0}{n}}+\ldots+e^{-\frac{(n-1)ST_0}{n}}\right]^{\gamma}B^{*}(ST_0/n)}{1+\sum\limits_{k=1}^{n}c_k\left(e^{-\frac{ST_0}{n}}\right)^{kn}}.$$

where $A^{*}(ST_0)$ and $B^{*}(ST_0/n)$ are polynomials in e^{ST_0} and $e^{ST_0/n}$ and the c_k are real coefficients. In the given case, $n = 2$ and $\gamma = 1$. Condition 2) requires that $A\gamma(ST_0) = 1$ and $c_k = 0$. In this case, we obtain

$$\Phi^{*}(ST_0)=e^{-ST_0}. \tag{16}$$

Since $D\left\{D_2^{-1}\left[\Phi^{*}\left(\frac{ST_0}{2}\right)\right]\right\}=\Phi^{*}(ST_0)$, then $B^{*}\left(\frac{ST_0}{2}\right)=e^{-\frac{ST_0}{2}}$ and

$$\Phi_1^*\left(S\frac{T_0}{2}\right)=\frac{\left(e^{\frac{ST_0}{2}}\right)^2-1}{\left(e^{\frac{ST_0}{2}}\right)^2\left(e^{\frac{ST_0}{2}}-1\right)}. \tag{17}$$

By substituting the corresponding terms, we get the schedule for computing the correcting device:

$$G^*\left(S\frac{T_0}{2}\right)=\frac{1.6e^{\frac{ST_0}{2}}-0.58}{e^{\frac{ST_0}{2}}-1}. \tag{18}$$

b) $n = 1$, $T_0 = 1$ second. In this case, (14) takes the following form:

$$G^*(ST_0)=\frac{1}{H^*(ST_0)}\,\frac{\Phi^*(ST_0)}{1-\Phi^*(ST_0)}.$$

The functions $\Phi^*(ST_0)$ and $H^*(ST_0)$ are easily determined:

$$\Phi^*(ST_0)=e^{-ST_0},\quad H^*(ST_0)=\frac{0.86}{e^{ST_0}-0.14}.$$

From the relationships given above, we obtain:

$$G^*(ST_0)=\frac{1.2e^{ST_0}-0.16}{e^{ST_0}-1}. \tag{19}$$

c) $n = 1$, $T_0 = 0.5$ seconds. Analogously to the previous case, we have

$$G^*\left(S\frac{T_0}{2}\right)=\frac{1.6e^{S\frac{T_0}{2}}-0.58}{e^{\frac{ST_0}{2}}-1}. \tag{20}$$

We remark that (20) coincides with (18).

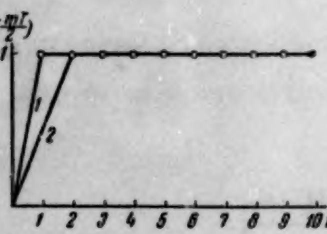

Fig. 6.

Curve 1 of Fig. 6 is the output function of the system $X(mT/2)$, where $\underline{m}$ is an integer, for unit external excitation, for the cases a) and c), and curve 2 is for case b).

It is clear, from the example given, that cases a) and c), from the point of view of rapidity of action, are equivalent (Cf. curve 1 on Fig. 6). Initial alignment will have the same transfer function for cases a) and c), bull will be simpler for case a) than for case c).

In the second example, we investigate the influence of the inequality of the pulse elements' repetition periods on system stability. We consider a static system of automatic control. We assume that, in this system,

$$G(S)=\frac{K}{S+a}\ \text{and}\ H(S)=\frac{1-e^{-\frac{ST}{n}}}{S}\,\frac{be^{-\tau S}}{S+b}.$$

We now find the maximum attainable value of system gain, $\overline{K}_{max}$, for $T_1 = T_0$ and $T_2 = T_0/n$ ($\underline{n}$ an integer). By means of Formula (10), the transfer function of the open-looped system can be determined:

$$\overline{K}W^*(ST_0)=e^{-(m_2-1)ST_0}\,\frac{\overline{K}(c_1e^{ST_0}+c_2)}{(e^{ST_0}-e^{-aT_0})(e^{ST_0}-e^{-bT_0})},$$

where

$$c_1 = \frac{\left[1 - e^{-(b-a)\frac{T_0}{n}} - e^{-b\Delta_1 \frac{T_0}{n}}\left(1 - e^{a\frac{T_0}{n}}\right)\right] e^{-a\Delta_2 T_0} + e^{-b\left(\frac{\Delta_1}{n} + \Delta_2\right)T_0}\left(1 - e^{b\frac{T_0}{n}}\right)}{e^{-a\frac{T_0}{n}} - e^{-b\frac{T_0}{n}}},$$

$$c_2 = \frac{\left[1 - e^{-(b-a)\frac{T_0}{n}} - e^{-b\Delta_1 \frac{T_0}{n}}\left(1 - e^{a\frac{T_0}{n}}\right)\right] e^{-(a\Delta_2 + b)T_0} + e^{-\left[a + b\left(\frac{\Delta_1}{n} + \Delta_2\right)\right]T_0}\left(1 - e^{b\frac{T_0}{n}}\right)}{e^{-a\frac{T_0}{n}} - e^{-b\frac{T_0}{n}}}.$$

To simplify the computations, we assume that $\tau = T_0$, i.e., $m_2 = 1$, $\Delta_1 = 0$ and $\Delta_2 = 1/n$. By substituting these values in the expression for c_2, we get

$$c_1 = 0, \quad c_2 = \frac{\left(1 - e^{-b\frac{T_0}{n}}\right)\left(e^{-aT_0} - e^{-bT_0}\right)}{e^{-a\frac{T_0}{n}} - e^{-b\frac{T_0}{n}}}. \tag{21}$$

Then, the transfer function of the open-looped system takes the form:

$$\overline{K}W^*(ST_0) = \frac{\overline{K}c_2}{(e^{ST_0} - e^{-aT_0})(e^{ST_0} - e^{-bT_0})}, \tag{22}$$

and the characteristic equation of the system is

$$e^{2ST_0} - (e^{-aT_0} + e^{-bT_0})e^{ST_0} + e^{-(a+b)T_0} + c_2\overline{K} = 0.$$

By employing the analog of the Routh-Hurwitz criterion, we determine the following stability conditions:

$$\begin{aligned} &1. \quad (1 - e^{-aT_0})(1 - e^{-bT_0}) + c_2\overline{K} > 0, \\ &2. \quad (1 - e^{(a+b)T_0}) - c_2\overline{K} > 0, \quad (23) \\ &3. \quad (1 + e^{-aT_0})(1 + e^{-bT_0}) + c_2\overline{K} > 0. \end{aligned} \tag{23}$$

Since c_2 is always positive for any positive values of $\underline{a}$ and $\underline{b}$, Conditions 1 and 3 always hold. Therefore, the condition for system stability will be

$$\overline{K}_{\max} < \frac{1 - e^{-(a+b)T_0}}{c_2}. \tag{24}$$

With given values of $\underline{a}$, $\underline{b}$ and T_0 it is possible, from this last inequality, to construct the functional relationship, $\overline{K}_{\max} = f(n)$ (where $\underline{n}$ is an integer). For intermediate points, it is necessary to use Formula (6), since inequality (23) will not be valid at these points. The following table gives values for c_2 and $\overline{K}_{\max}$, computed from Formulae (21) and (23), for various values of $\underline{n}$, for a = 4, b = 2 and T_0 = 1 second.

n	1	2	3	4	5	6	7	8	9	10	∞
c_2	0.87	0.32	0.22	0.19	0.18	0.16	0.155	0.150	0.145	0.140	0.117
$\overline{K}_{\max}$	1.16	3.18	4.50	5.16	5.65	6.16	6.41	6.76	6.90	7.10	8.55

The curve for $\overline{K}_{\max} = f(n)$ is given in Fig. 7.

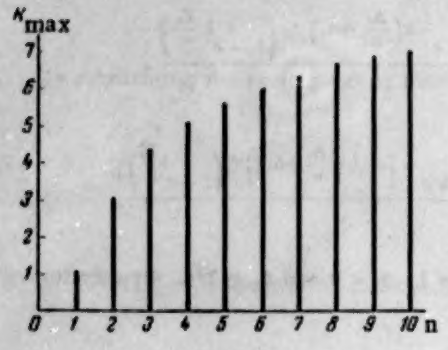

Fig. 7.

It is clear from these examples that, with increasing n, i.e., with a decreasing repetition period of the second pulse element, the stability margin of the system increases in comparison with the case $T_1 = T_2$.

4. A Method for Constructing the Transient Response in Pulse Systems Containing Two Pulse Elements

The transient response can be determined analytically by means of the formulae for inverting the Laplace transform, or constructed by graphic or tabular fitting.

We investigate both these methods.

1. We use the formula for inverting the discrete Laplace transform [2]:

$$X[nT_1, \varepsilon] = \frac{1}{2\pi j} \int_{c-j\frac{\pi}{\omega_1}}^{c+j\frac{\pi}{\omega_1}} X^*(ST_1, \varepsilon)\, e^{S(n+\varepsilon)T_1}\, dS = \frac{1}{2\pi j} \oint_{\Gamma_1} X^*(ST_1, \varepsilon)\, e^{S(n+\varepsilon)T_1}\, dS. \tag{25}$$

Here, n is an integer and the integration is performed along the contour Γ_1, bounded by the lines corresponding to c and $\pm j\pi/\omega_1$. The function $X^*(ST_1, \varepsilon)$ occurring in (25) can be found by applying the discrete Laplace transform according to Formula (2):

$$X^*(ST_1, \varepsilon) = E^*(ST_1)\frac{\gamma_1\gamma_2}{T_1T_2} \sum_{m=-\infty}^{\infty} \sum_{l=-\infty}^{\infty} G(S + jm\omega_1)\, H(S + jl\omega_2 + jm\omega_1) \times$$
$$\times\, e^{jlm\frac{\omega_2}{\omega_1}2\pi} e^{jm\omega_1\varepsilon T_1}.$$

From these last two decompositions we obtain:

$$X[nT_1, \varepsilon] = \frac{1}{2\pi j} \oint_{\Gamma_1} \frac{F^*(ST_1)\frac{\gamma_1\gamma_2}{T_1T_2} \sum\limits_{m=-\infty}^{\infty} \sum\limits_{l=-\infty}^{\infty} G(S + jm\omega_1) H(S + jl\omega_2 + jm\omega_1)\, e^{jlm\frac{\omega_2}{\omega_1}2\pi}}{1 + \frac{\gamma_1\gamma_2}{T_1T_2} \sum\limits_{m=-\infty}^{\infty} \sum\limits_{l=-\infty}^{\infty} G(S + jm\omega_1)\, H(S + jl\omega_2 + jm\omega_1)\, e^{jlm\frac{\omega_2}{\omega_1}2\pi}} \times \tag{26}$$
$$\times\, e^{jm\omega_1\varepsilon T_1} e^{S(n+\varepsilon)T_1} dS.$$

In the general case it is possible, in principle, to obtain the transient response by using the last decomposition. However, the integration can be simplified if approximate Formula (6) is used.

2. It is clear from Fig. 2 that

$$X(S) = C^*(ST_2)\, H(S) = \sum_{l=-\infty}^{\infty} c(lT_2)\, e^{-lT_2S} HS,$$

$$C(S) = E^*(ST_1)\, G(S) = \sum_{m=-\infty}^{\infty} e(mT_1)\, e^{-mT_1S} G(S),$$

where

$$c(lT_2) = c(t)\big|_{t=lT_2} = L^{-1}[C(S)]\big|_{t=lT_2},$$

$$e(mT_1) = e(t)\big|_{t=mT_1} = L^{-1}[E(S)]\big|_{t=mT_1}.$$

By using the formula for the inverse Laplace transformation, we obtain

$$x(t) = \sum_{l=-\infty}^{\infty} c(lT_2)\, h(t-lT_2),$$

$$c(t) = \sum_{m=-\infty}^{\infty} e(mT_1)\, g(t-mT_1).$$

From these last two expressions we get

$$x(t) = \sum_{l=0}^{\infty} \sum_{m=0}^{nT_2/T_1} e(mT_1)\, g(nT_2 - mT_1)\, h(t-lT_2).$$

The last expression can be given in the following expanded form:

$$\begin{aligned} x(t) &= h(t)\,[e(0)\,g(0)] + \\ &+ h(t-T_2)\,[e(0)\,g(T_2) + e(T_1)\,g(T_2-T_1) + e(2T_1)\,g(T_2-2T_1) + \ldots] + \\ &+ h(t-2T_2)\,[e(0)g(2T_2) + e(T_1)g(2T_2-T_1) + e(2T_1)g(2T_2-2T_1) + \ldots] + \\ &+ \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots + \\ &+ h(t-KT_2)\,]e(0)g(KT_2) + e(T_1)g(KT_2-T_1) + e(2T_1)g(KT_2-2T_1) + \ldots + \\ &+ \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \end{aligned} \tag{27}$$

Such is the method of attacking ordinary pulse systems expounded in [2, 4].

If the external forcing function $f(t)$ is given, and g(t) and h(t) are known then, by using Formula (27), it is possible to construct the system's transient response step by step.

Appendix

We assume that G(S) and H(S) have the form:

$$G(S) = \sum_j \frac{c_j}{S+a_j}, \quad H(S) = \sum_j \frac{d_j}{S+b_i} e^{-\tau S},$$

where τ is a constant value of the object lag. Then, we can find $\overline{K}W^*(ST_0) = D\,\{D_n^{-1}[G^*(ST_0/n)H^*(ST_0/n)]\}$ in the general form. Figure 8 shows a typical transient response $h_1(t)$. We assume that $\tau = (m_1 - \Delta_1)(T_0/n)$, where m_1 is an integer and $0 \leq \Delta_1 < 1$. We find $G^*(ST_0/n)$ and $H^*(ST_0/n)$:

$$G^*\left(S\frac{T_2}{n}\right) = \gamma_1 \sum_j \frac{c_j e^{S\frac{T_0}{n}}}{e^{S\frac{T_0}{n}} - e^{-a_j\frac{T_0}{n}}},$$

$$H^*\left(S\frac{T_0}{n}\right) = \gamma_2 e^{-S\frac{T_0}{n}m_1} \sum_i \frac{d_i e^{S\frac{T_0}{n}} e^{-b_i\Delta_1\frac{T_0}{n}}}{e^{S\frac{T_0}{n}} - e^{-b_i\frac{T_0}{n}}}.$$

We use the notation:

$$\overline{KW}^*\left(S\frac{T_0}{n}\right)=G^*\left(S\frac{T_0}{n}\right)H^*\left(S\frac{T_0}{n}\right)=$$

$$=\gamma_1\gamma_2 e^{-S\frac{T_0}{n}m_1}\sum_j\sum_i \frac{c_jd_i e^{S\frac{T_0}{n}} e^{-h_i\Delta_1\frac{T_0}{n}}}{e^{-a_j\frac{T_0}{n}}-e^{-b_i\frac{T_0}{n}}}\left[\frac{e^{S\frac{T_0}{n}}}{e^{S\frac{T_0}{n}}-e^{-a_j\frac{T_0}{n}}}-\frac{e^{S\frac{T_0}{n}}}{e^{S\frac{T_0}{n}}-e^{-b_i\frac{T_0}{n}}}\right].$$

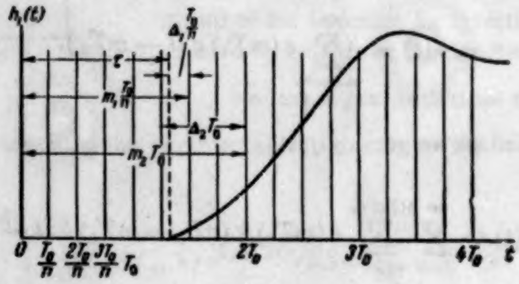

Fig. 8.

Knowing $\overline{KW}^*(ST_0/n)$, we can find the corresponding function:

$$\overline{KW}(S)=\gamma_2\sum_j\sum_i \frac{c_jd_i e^{-b_i\Delta_1\frac{T_0}{n}}}{e^{-a_j\frac{T_0}{n}}-e^{-b_i\frac{T_0}{n}}}\left[\frac{e^{-(m_1-1)\frac{T_0}{n}S}}{S+a_j}-\frac{e^{-(m_1-1)\frac{T_0}{n}S}}{S+b_i}\right].$$

Since $D\{D_n^{-1}[\overline{KW}^*(ST_0/n)]\}=\overline{KW}^*(ST_0)$, we can from this, obtain the generalized formula for $\overline{KW}^*(ST_0)$ (Cf. Fig. 4):

$$\overline{KW}^*(ST_0)=\gamma_1\gamma_2\sum_j\sum_i \frac{c_jd_i e^{-b_i\Delta_1\frac{T_0}{n}} e^{-m_2ST_0}}{e^{-a_j\frac{T_0}{n}}-e^{-b_i\frac{T_0}{n}}}\times \tag{28}$$

$$\times\left[\frac{e^{ST_0}e^{-a_j\Delta_2T_0}}{e^{ST_0}-e^{-a_jT_0}}-\frac{e^{ST_0}e^{-b_i\Delta_2T_0}}{e^{ST_0}-e^{-b_iT_0}}\right],$$

where $m_2=(m_1-1)/n+\Delta_2$, m_2 is an integer and $0\le\Delta_2<1$ (Fig. 8).

In the case $\tau=0$ (i.e., $m_1=m_2=0$, $\Delta_2=1/n$), the last decomposition assumes the following form:

$$\overline{KW}^*(ST_0)=\gamma_1\gamma_2\sum_j\sum_i \frac{c_jd_i}{e^{-a_j\frac{T_0}{n}}-e^{-b_i\frac{T_0}{n}}}\left[\frac{e^{ST_0}e^{-a_j\frac{T_0}{n}}}{e^{ST_0}-e^{-a_jT_0}}-\frac{e^{ST_0}e^{-b_i\frac{T_0}{n}}}{e^{ST_0}-e^{-b_iT_0}}\right]. \tag{29}$$

We now consider the following particular cases of the last expression.

1. For $n=\infty$ (i.e., $T_2=0$), Decomposition (29) takes the following form:

$$\overline{KW}^*(ST_0)=\gamma_1\sum_j\sum_i \frac{c_jd_i}{b_i-a_j}\left[\frac{e^{ST_0}}{e^{ST_0}-e^{-a_jT_0}}-\frac{e^{ST_0}}{e^{ST_0}-e^{-b_iT_0}}\right].$$

2. For n = 1 (i.e., $T_1 = T_2 = T_0$), it will be

$$\overline{K}W^*(ST_0) = \gamma_1\gamma_2 \sum_j \sum_i \frac{c_j d_i e^{ST_0}}{e^{-a_j T_0} - e^{-b_i T_0}} \left[\frac{e^{-a_j T_0}}{e^{ST_0} - e^{-a_j T_0}} - \frac{e^{-b_i T_0}}{e^{ST_0} - e^{-b_i T_0}} \right].$$

Expressions (29) and (28) are also valid in the case when there is one pole at the origin of coordinates in G(S) or H(S). If some poles of G(S) and H(S) coincide, i.e.,

$$G(S) = \sum_\lambda \frac{c_\lambda}{S + a_\lambda} + \sum_j \frac{c_j}{S + a_j}$$

and

$$H(S) = \sum_\lambda \frac{d_\lambda}{S + a_\lambda} e^{-\tau S} + \sum_i \frac{d_i}{S + b_i} e^{-\tau S},$$

then in this case $\overline{K}W^*(ST_0/n)$ can be given, as previously, as a sum of two terms:

$$\overline{K}W^*\left(S\frac{T_0}{n}\right) = \overline{K}W_1^*\left(S\frac{T_0}{n}\right) + \overline{K}W_2^*\left(S\frac{T_0}{n}\right) = \gamma_1\gamma_2 n \sum_\lambda \frac{c_\lambda d_\lambda e^{2S\frac{T_0}{n}} e^{-S\frac{T_0}{n}m_1} e^{-a_\lambda \Delta_1 \frac{T_0}{n}}}{\left(e^{S\frac{T_0}{n}} - e^{-a_\lambda \frac{T_0}{n}}\right)^2} +$$

$$+ \gamma_1\gamma_2 \left[\sum_j \sum_i \frac{c_j d_i e^{2S\frac{T_0}{n}} e^{-S\frac{T_0}{n}m_1} e^{-b_i \Delta_1 \frac{T_0}{n}}}{\left(e^{S\frac{T_0}{n}} - e^{-a_j \frac{T_0}{n}}\right)\left(e^{S\frac{T_0}{n}} - e^{-b_i \frac{T_0}{n}}\right)} + \sum_i \sum_\lambda \frac{c_\lambda d_i e^{2S\frac{T_0}{n}} e^{-S\frac{T_0}{n}m_1} e^{-b_i \Delta_1 \frac{T_0}{n}}}{\left(e^{S\frac{T_0}{n}} - e^{-a_\lambda \frac{T_0}{n}}\right)\left(e^{S\frac{T_0}{n}} - e^{-b_i \frac{T_0}{n}}\right)} + \right.$$

$$\left. + \sum_j \sum_\lambda \frac{c_j d_\lambda e^{2S\frac{T_0}{n}} e^{-S\frac{T_0}{n}} e^{-a_\lambda \Delta_1 \frac{T_0}{n}}}{\left(e^{S\frac{T_0}{n}} - e^{-a_j \frac{T_0}{n}}\right)\left(e^{S\frac{T_0}{n}} - e^{-a_\lambda \frac{T_0}{n}}\right)} \right],$$

where $\overline{K}W_2^*(ST_0/n)$ is a sum of products of terms with nonidentical poles. The form of each of these products coincides with Expression (28), so that is necessary to consider only $\overline{K}W_1^*(ST_0/n)$:

$$\overline{K}W_1^*\left(S\frac{T_0}{n}\right) = \gamma_1\gamma_2 n \sum_\lambda \frac{c_\lambda d_\lambda e^{2S\frac{T_0}{n}} e^{-S\frac{T_0}{n}m_1 - a_\lambda \Delta_1 \frac{T_0}{n}}}{\left(e^{S\frac{T_0}{n}} - e^{-a_\lambda \frac{T_0}{n}}\right)^2}.$$

We find from this last expression that

$$\overline{K}W_1(S) = \gamma_2 \sum_\lambda \frac{c_\lambda d_\lambda e^{-a_\lambda \Delta_1 \frac{T_0}{n}} e^{-(m_1 - 1)\frac{T_0}{n}S}}{e^{-a_\lambda \frac{T_0}{n}} (S + a_\lambda)^2}.$$

Knowing the expression for $\overline{K}W_1(S)$, we can easily find

$$\overline{K}W_1^*(ST_0) = \gamma_1\gamma_2 n \sum_\lambda \frac{c_\lambda d_\lambda e^{-a_\lambda \Delta_1 \frac{T_0}{n}} e^{(1-m_2)ST_0} e^{ST_0} e^{-a_\lambda \Delta_2 T_0} [e^{-a_\lambda T_0} + \Delta_2 (e^{ST_0 - a_\lambda T_0})]}{e^{-a_\lambda \frac{T_0}{n}} \left(e^{ST_0} - e^{-a_\lambda T_0}\right)^2}.$$

For $\tau = 0$, the last decomposition takes the following form:

$$\overline{K}W_1^*(ST_0) = \gamma_1\gamma_2 n \sum_{\lambda} \frac{c_\lambda d_\lambda e^{ST_0}\left[e^{-a_\lambda T_0} + \frac{1}{n}\left(e^{ST_0} - e^{-a_\lambda T_0}\right)\right]}{\left(e^{ST_0} - e^{-a_\lambda T_0}\right)^2}.$$

It is easily seen that this last expression also holds in the case when one of the roots a_λ is zero.

SUMMARY

1. The transfer function of a pulse system containing two pulse elements with unequal repetition periods was found.

2. By using the given simple conditions for a correct choice of the ratio of T_1 and T_2, one can increase the stability margin of the system. In general, the condition $T_1 = T_2$ is not the best regimen for system functioning from the point of view of increased stability margin.

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DETERMINATION OF PERIODIC BEHAVIOR OF AUTOMATIC CONTROL SYSTEMS CONTAINING A NONLINEAR ELEMENT WITH BROKEN-LINE CHARACTERISTIC

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A method is considered of determining the periodic behavior in such automatic control systems which differ from the linear ones by the presence of an arbitrary characteristic of broken-line type. The periodic behavior is fully described by complete Fourier series, without neglecting higher harmonics. The problem is reduced to the solving of $\underline{h}$ simultaneous transcendental equations (equations of periods) which determine times of motion in each part of the nonlinear characteristic within the period limits. Also some problems connected with the use of computers for solving period equations are considered at the end of the paper.

Self-oscillatory behavior is characteristic of some automatic control systems. Consequently it is often necessary to determine the period and the form of self-oscillations from the known equations of all elements of the control system. In systems having vibrators, or in systems subjected to periodical perturbations, there arises the problem of determining the period and the form of the forced oscillations in nonlinear systems. An approximate answer to these questions, for example, by harmonic balance method, often proves unsatisfactory as the automatic control systems do not by any means satisfy everywhere the conditions under which such methods are applicable. In such cases the periodic behavior must be calculated exactly, and this always involves considerable difficulties. Even when one succeeds in separating one "principal" nonlinearity in the system, and in reducing the problem to the study of an automatic control system which differs from a linear one only by the presence of one nonlinear element, even then the exact solution can only be obtained on the assumption that the characteristic of the equation is of a broken-line type. The problem is reduced to the solving of period equations, that is, of final – even if transcendental – equations whose roots, while satisfying some additional conditions, determine directly the required periodic behavior. There are two methods described in the literature which permit exact determination of the periodic behavior in this broken-line case, that is, permit the construction of period equations.

1. <u>"Make to fit" method.</u> In applying this method it is necessary to integrate out all equations of the linear systems into which the nonlinear system under consideration has been decomposed, and the periodic solutions are constructed by using these integrals.

2. <u>Finding periodic solutions in complete Fourier-series form (not neglecting higher harmonics).</u> In this case it is not necessary to integrate the corresponding linear equations, and the period equations are directly expressed by coefficients of original simultaneous equations which describe the motion of the automatic control system under consideration. The problem which has been solved by this method was a particular one when the characteristic of the nonlinear element consisted of finite straight lines parallel to two given straight lines only [1].

The method is applied to more general systems in this paper, that is, to systems with an arbitrary broken-line characteristic (the number of different directions of portions of the characteristics is now arbitrary). As distinct from [1], the affine transformation of the given characteristic into a characteristic consisting of straight lines parallel to coordinate axes is not applied here. This brings about a considerable simplification in the equations which serve to determine the times of motion of a representative point in each separate portion of the characteristic.

If computors are available it is convenient to apply the proposed algorithm when there are a few component portions of the characteristic and the linear part of the system is of a high order. In a system with a low-order linear part it is obviously simpler to have a model of the problem and "to get the feel" of the phase space, and thus detect the stable limit cycles.

The stability of periodic behavior is not discussed in this paper. These problems are considered in [2].

1. Introductory Relations

Our problem can be mathematically formulated as follows. We have simultaneous differential equations

$$\frac{d\bar{x}_i}{dt} = \sum_{j=1}^{n} \alpha_{ij}\bar{x}_j + \varkappa_i y, \quad y = f(\bar{x}_1) \quad (i = 1, 2, \ldots, n), \tag{1}$$

where α_{ij} and κ_i are real, and $y = f(\bar{x}_1)$ is any broken-line function, whether continuous or discontinuous.

The transition from one portion of the characteristic $y = f(\bar{x}_1)$ to another, is said to be taking place normally in the sense of [3] when sliding or ambiguous transitions are absent. The type of periodical solution, that is, the sequence of successive straight lines which form the characteristic $y = f(\bar{x}_1)$, is given for the duration of periodic motion. The problem is then reduced to the solving of the final period equations which determine the times taken by each of these portions.

Eliminating from simultaneous Equations (1) all $\bar{x}_i$ except $\bar{x}_1$, we obtain equation

$$D(p)\bar{x} = K(p)f(\bar{x}) \quad (p = d/dt), \tag{2}$$

which we shall call the derived equation (to simplify notation, subscript 1 of $\bar{x}_1$ shall be omitted from now on). In addition, for every moment of time t_q, during which a break occurs in any function $\bar{x}(t)$ or function $y = f(\bar{x})$ or in any derivatives up to order $(n-1)$ inclusive, we obtain gap relations (see [4]):

$$\begin{aligned} a_0\xi_0 &= b_0\eta_0, \\ a_0\xi_1 + a_1\xi_0 &= b_0\eta_1 + b_1\eta_0, \\ &\ldots\ldots\ldots\ldots \\ a_0\xi_{n-1} + \ldots + a_{n-1}\xi_0 &= b_0\eta_{n-1} + \ldots + b_{n-1}\eta_0. \end{aligned} \tag{3}$$

In (2) and (3)

$$D(p) = a_0p^n + a_1p^{n-1} + \ldots + a_n$$
$$K(p) = b_0p^n + b_1p^{n-1} + \ldots + b_n$$

are linear differential operators,* and

$$\begin{aligned} \xi_k &= \bar{x}^{(k)}(t_q + 0) - \bar{x}^{(k)}(t_q - 0), \\ \eta_k &= y^{(k)}(t_q + 0) - y^{(k)}(t_q - 0) \end{aligned} \tag{4}$$

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*The degree of K(p) is less than that of D(p) but we regard them as equal by assuming that the highest coefficients in $b_0, b_1, \ldots, b_s$ are equal to zero.

are the heights of gaps of the k-th derivative ($k = 0, 1, \ldots, n-1$) of functions $\bar{x}(t)$ and $y(t)$, respectively, at the time $t = t_q$.

Further, quantities ξ_k and η_k are considered bounded, that is, no gaps of the second type occur either in the functions or in their derivatives.

When the equation of the investigated straight-line piece of the characteristic with subscript q is written in the form $y = k_q\bar{x} + \lambda_q$ and then this y is substituted back into (2), we obtain

$$L_q(p)\,\bar{x} = c_q \quad (q = 1, 2, \ldots, h), \tag{5}$$

where the notation $L_q(p) = D(p) - k_qK(p) = g_0^q p^n + g_1^q p^{n-1} + \ldots + g_n^q$, $c_q = b_n\lambda_q$, $g_r^q = a_r - k_qb_r$.

The pattern of the required periodic behavior determines the sequence of the successive differential equations of the type of (5). Some equations in this sequence, though corresponding to different subscripts, may be identical, due to the fact that during the periodic motion one and the same portions of the characteristic can occur more than once.

Therefore to solve our problem it is necessary to find a periodic function $\bar{x}(t)$ with period $T = t_h$ and satisfying, successively, the equations

$$\begin{aligned} &L_1(p)\,\bar{x}(t) = c_1 \text{ when } 0 < t < t_1,\\ &L_2(p)\,\bar{x}(t) = c_2 \text{ when } t_1 < t < t_2,\\ &\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\\ &L_h(p)\,\bar{x}(t) = c_h \text{ when } t_{h-1} < t < t_h. \end{aligned} \tag{6}$$

Then the gaps of the functions $\bar{x}(t)$, $y = f[\bar{x}(t)]$ and of their $n-1$ derivatives should satisfy the gap relations (3) at each point t_q ($q = 1, 2, \ldots, h$). The numbers $t_1, t_2, \ldots, t_h$ are unknown and must also be determined.

2. Transforming Equations and Gap Relations

We shall find the periodic solution of Eq. (6) and (3) in the form

$$\bar{x}(t) = x(t) + \sum_{q=1}^{h} \rho_q\,[1(t - t_{q-1}) - 1(t - t_q)], \tag{7}$$

where $x(t) = \sum_{q=1}^{h} x_q(t)$, $1(t)$ is the Heaviside unit function, $\rho_q = \frac{c_q}{g_n^q}$, and $x_q(t)$ is an auxiliary periodic function, with period T, and satisfying the following conditions:

$$\begin{aligned} x_q(t) &= 0 \quad \text{when } 0 < t < t_{q-1},\\ L_q(p)\,x_q(t) &= 0 \quad \text{when } t_{q-1} < t < t_q,\\ x_q(t) &= 0 \quad \text{when } t_q < t < t_h. \end{aligned} \tag{8}$$

The sum on the right-hand side of (7) represents a step function of t, taking the values ρ_q in the intervals $t_{q-1} < t < t_q$.

For a particular interval with subscript q the transition from $\bar{x}(t)$ to $x(t)$ in accordance with (7) is in the form

$$\bar{x}(t) = x(t) + \rho_q \quad (t_{q-1} < t < t_q). \tag{9}$$

Therefore the q-th equation of the System (6), valid in the same interval, takes the form

$$L_q(p)\,\bar{x}(t) = L_q(p)\,[x(t) + \rho_q] = L_q(p)\,x(t) + g_n^q\rho_q = c_q.$$

But $\rho_q = c_q/g_n^q$ and when $t_{q-1} < t < t_q$, x(t) is determined by the homogeneous equation

$$L_q(p)\,x(t) = 0 \qquad (q = 1, 2, \ldots, h). \tag{10}$$

Introducing the notation

$$\begin{aligned} x^{(k)}(t_q + 0) &= x^k_{q2}, & y^{(k)}(t_q + 0) &= y^k_{q2}, \\ x^{(k)}(t_q - 0) &= x^k_{q1}, & y^{(k)}(t_q - 0) &= y^k_{q1}, \end{aligned} \tag{11}$$

and taking into consideration the equation of the characteristic $y = f(\overline{x})$ and Relations (7), we are able to transform the gap relations. As a result we obtain a system of $\underline{n}$ linear equations (deduced in Appendix I):

$$\begin{aligned} x^0_{q1} &= u_{q0}x^0_{q2} - v_{q0}, \\ x^1_{q1} &= u_{q1}x^0_{q2} + u_{q0}x^1_{q2} - v_{q1} \\ &\cdots\cdots\cdots\cdots\cdots\cdots \\ x^{n-1}_{q1} &= u_{q,\,n-1}x^0_{q2} + u_{q,\,n-2}\,x^1_{q2} + \ldots + u_{q0}x^{n-1}_{q2} - v_{q,\,n-1}\,. \end{aligned} \tag{12}$$

Here we have introduced the notation $u_{q0} = g_0^{q+1}/g_0^q$, $v_{q0} = \psi_{q0}/g_0^q$. All the other u_{qk} and v_{qk} are given by recurrence formulas

$$u_{qk} = \frac{g_k^{q+1}}{g_0^q} - \sum_{i=1}^{k} \frac{g_i^q}{g_0^q}\,u_{q,\,k-i}\,, \qquad v_{qk} = \frac{\psi_{qk}}{g_0^q} - \sum_{i=1}^{k} \frac{g_i^q}{g_0^q}\,v_{q,\,k-i}\,, \tag{13}$$

where $\psi_{qk} = \rho_{q+1}(b_k - g_k^{q+1}) - \rho_q(b_k - g_k^q)$.

We now remember that b_k and g_k^q are the coefficients of the polynomials K(p) and $L_q(p)$. As all the functions considered here and below are periodic, we are allowed to change subscript h + 1 simply into 1.

We shall call System (12) the cross-over condition, from the q-th to the (q + 1)th piece of the characteristic.

Introducing column matrix

$$X(t) = \begin{Vmatrix} x(t) \\ \dot{x}(t) \\ \cdot \\ \cdot \\ x^{(n-1)}(t) \end{Vmatrix}, \quad X(t_q + 0) = X_{q2} = \begin{Vmatrix} x^0_{q2} \\ x^1_{q2} \\ \cdot \\ \cdot \\ x^{n-1}_{q2} \end{Vmatrix},$$

$$X(t_q - 0) = X_{q1} = \begin{Vmatrix} x^0_{q1} \\ x^1_{q1} \\ \cdot \\ \cdot \\ x^{n-1}_{q1} \end{Vmatrix} \tag{14}$$

we write (12) in matrix form:

$$X_{q1} = U_q X_{q2} - V_q \qquad (q = 1, 2, \ldots, h). \tag{15}$$

In (15) U_q is a triangular truncated-rows matrix: *

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* We are able to evaluate det U_q by making use of the footnote on p. 912 From either $u_{q0} = g_0^{q+1}/g_0^q = a_0/a_0 = 1$, or $b_0 = 0$ we have det $U_q = 1$. Therefore the matrix U_q is nondegenerate. $k_q = \infty$ could be an exceptional case, but then as long as the motion is in the portion corresponding to k_q, $\overline{x}$ = const, that is, it is known.

$$U_q = \left\| \begin{matrix} u_{q0} & & 0 \\ u_{q1} & u_{q0} & 0 \\ u_{q2} & u_{q1} & \\ \cdot & \cdot & \cdot \quad \cdot \\ u_{q,\,n-1} & & u_{q0} \end{matrix} \right\|,$$

and V_q is the column matrix

$$V_q = \left\| \begin{matrix} v_{q0} \\ v_{q1} \\ \cdot \\ \cdot \\ \cdot \\ v_{q,\,n-1} \end{matrix} \right\|.$$

Thus, the previously formulated question has been reduced to the determination of a discontinuous periodic function x(t) satisfying, successively, Equations (10) and cross-over Conditions (12) or (15) at every point of discontinuity $t = t_q$.

3. Construction of Auxiliary Function $x_q(t)$

We are introducing periodic functions $R_{kq}(t)$ ($k = 0, 1, \ldots, n-1$), period $T = t_h$, satisfying the following conditions:

1. Function $R_{kq}(t)$ is a solution of the differential equation $L_q(p)R_{kq}(t) = 0$ for all $\underline{t}$ except $t = \nu T$ ($\nu = -\infty, \ldots, -2, -1, 0, 1, 2, \ldots, \infty$).
2. All derivatives $R_{kq}^{(j)}(t)$ when $0 \le j < k$ are continuous.
3. k-th order derivative $R_{kq}^{(k)}(t)$ is discontinuous at points νT; the magnitude of its gap is unity.
4. All derivatives $R_{kq}^{(j)}(t)$ when $k < j \le n-1$ are again continuous.*

The construction of functions $R_{kq}(t)$ is shown in Appendix II, and it is also shown there that their Fourier-series expansion is of the form

$$R_{kq}(t) = \frac{1}{T} \sum_{r=-\infty}^{\infty} \frac{l_k^q(s)}{L_q(s)} e^{st}, \tag{16}$$

where $s \equiv ir\omega \left(\omega = \frac{2\pi}{T} \right)$, and

$$l_k^q(s) = g_{n-k-1}^q + g_{n-k-2}^q s + \ldots + g_0^q s^{n-k-1}.$$

Making use of the obtained functions $R_{kq}(t)$ we construct the periodic function $x_q(t)$ defined by the expression

$$x_q(t) = \sum_{k=0}^{n-1} R_{kq}(t - t_{q-1})\alpha_k - \sum_{k=0}^{n-1} R_{kq}(t - t_q)\beta_k, \tag{17}$$

where α_k and β_k are arbitrary real numbers and $0 \le t_{q-1} < t_q \le T$.

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*The discontinuous function $R_{kq}^{(k)}(t)$ is not differentiable at the points of discontinuity, and the continuity of its derivatives is understood here in the sense of formal continuity, that is, the left-hand and right-hand side derivatives are equal to one another at every point of discontinuity νT.

On the basis of the properties of functions $R_{kq}(t)$ one is able to draw the following conclusions regarding the properties of $x_q(t)$:

a) $x_q(t)$ is a solution of the differential equation $L_q(p)x_q(t) = 0$ for all $\underline{t}$ with the exception of points $t_{q-1} + \nu T$ and $t_q + \nu T$ (integral ν) and, therefore, the function and its first $n-1$ derivatives are continuous everywhere except at these points;

b) within period T the gap of the k-th derivative of the function $x_q(t)$ at the point t_{q-1} is equal to α_k:

$$x_q^k(t_{q-1}+O) - x_g^k(t_{q-1}-O) = \alpha_k.$$

Similarly, at the point t_q we have

$$x_q^{(k)}(t_q+O) - x_q^{(k)}(t_q-O) = -\beta_k.$$

Having evaluated $\dot{x}_q(t), \ddot{x}_q(t), \ldots, x_q^{(n-1)}(t)$ from (17) (the evaluation method of the derivatives of function $R_{kq}(t)$ is described below), and having formed the column matrices

$$X_q(t) = \left\|\begin{matrix} x_q(t) \\ \dot{x}_q(t) \\ \cdot \\ \cdot \\ x_q^{(n-1)}(t) \end{matrix}\right\|, \quad A = \left\|\begin{matrix} \alpha_0 \\ \alpha_1 \\ \cdot \\ \cdot \\ \alpha_{n-1} \end{matrix}\right\| \quad B = \left\|\begin{matrix} \beta_0 \\ \beta_1 \\ \cdot \\ \cdot \\ \beta_{n-1} \end{matrix}\right\|,$$

we are able to write the obtained equalities in matrix form:

$$X_q(t) = M_q(t-t_{q-1})A - M_q(t-t_q)B, \tag{18}$$

where $M_q(t)$ is the matrix

$$M_q(t) = \left\|\begin{matrix} R_{0q}(t) & R_{1q}(t) & \ldots R_{n-1,q}(t) \\ R'_{0q}(t) & R'_{1q}(t) & \ldots R'_{n-1,q}(t) \\ \cdot & \cdot & \cdot \\ R_{0q}^{(n-1)}(t) & R_{1q}^{(n-1)}(t) & \ldots R_{n-1,q}^{(n-1)}(t) \end{matrix}\right\|.$$

Now it is required that the function $x_q(t)$ defined by (17) be identically zero in the intervals $(0, t_{q-1})$ and (t_q, T), that is, that it be the trivial solution of equation $L_q(p)x_q(t) = 0$ in these intervals. It is obviously sufficient for it to satisfy conditions $x_q(t) = \dot{x}_q(t) = \ldots = x_q^{(n-1)}(t) = 0$, or in other words, $X_q(t) = 0$ for any $\underline{t}$ from these intervals. In particular, having taken $\underline{t}$ in the interval (t_q, T), we obtain $x_q(t) \equiv 0$ not only inside interval (t_q, T) but also inside interval $(0, t_{q-1})$ because function $x_q(t)$, together with its $n-1$ derivatives, is continuous at points 0 and T, according to property (a).

We select a point $t_q + 0$ in the interval (t_q, T). At this point the requirement $X_q(t_q + 0) = 0$ can be written in the form

$$M_q(t_q - t_{q-1})A - M_q(+O)B = 0. \tag{19}$$

in agreement with (18).

The values of A and B satisfying (19) will now be considered. We have then

$$X_q(t_{q-1}-O) = X_q(t_q+O) = 0. \tag{20}$$

From property (b) of function $x_q(t)$ we have

$$X_q(t_{q-1}+O) - X_q(t_{q-1}-O) = A, \quad X_q(t_q+O) - X_q(t_q-O) = -B. \tag{21}$$

It follows from (20) and (21)

$$X_q(t_{q-1}+O)=A, \quad X_q(t^q-O)=B.$$

The choice of A and B satisfying (19) ensures that they are equal to column matrices of initial and final values of the function $x_q(t)$, (an integral curve of the equation), and of its n −1 derivatives at both ends of the interval (t_{q-1}, t_q). Therefore one can substitute for column matrices A and B the column matrices $X_q(t_{q-1}+0)$ and $X_q(t_q-0)$ in (19) and (18). Having performed the substitution, we obtain

$$M_q(t_q-t_{q-1})X_q(t_{q-1}+O)-M_q(+O)X_q(t_q-O)=0, \tag{22}$$

$$X_q(t)=M_q(t-t_{q-1})X_q(t_{q-1}+O)-M_q(t-t_q)X_q(t_q-O), \tag{23}$$

$$x_q(t)=\sum_{k=0}^{n-1}R_{kq}(t-t_{q-1})x_q^{(k)}(t_{q-1}+O)-\sum_{k=0}^{n-1}R_{kq}(t-t_q)x_q^{(k)}(t_q-O). \tag{24}$$

Formulas (23) and (24) together with (22) determine the periodic function $x_q(t)$, period $T = t_h$, satisfying Conditions (8).

4. Period Equations

We are similarly constructing functions $x_q(t)$ in each interval (t_{q-1}, t_q) $(q = 1, 2, \dots, h)$. The sum $x(t) = \sum_q x_q(t)$ [and correspondingly $X(t) = \sum_q X_q(t)$] complies with (10).

Now it is necessary that the gap Conditions (15) be fulfilled at each point t_q. Taking first into consideration the obvious equalities [see notations (11) and (14)]

$$X(t_q-0)=X_{q1}=X_q(t_q-O),$$
$$X(t_q+0)=X_{q2}=X_{q+1}(t_q+O),$$
$$X(t_{q-1}+0)=X_{q-1,2}=X_q(t_{q-1}+O)$$

and then making use of (15) we can put (22) in the following form:

$$M_q(t_q-t_{q-1})X_{q-1,2}-M_q(+O)U_qX_{q2}-M_q(+O)V_q=0. \tag{25}$$

Relation (25), when the value of x(t) and its derivatives at the point $t_{q-1}+0$ is known, permits to determine the values of x(t) together with its derivatives at the point t_q+0.

Taking into account the gap conditions, formula (23) becomes

$$X_q(t)=M_q(t-t_{q-1})X_{q-1,2}-M_q(t-t_q)U_qX_{q2}-M_q(t-t_q)V_q. \tag{26}$$

By adding to the system of Equations (25) the condition $X_{h2} = X_{02}$, which should be true in view of the periodicity of the required solution x(t), a closed system of equations is obtained:

$$\begin{gathered}M_1(t_1)X_{02}-N_1(+O)X_{12}=S_1(+O),\\ M_2(t_2-t_1)X_{12}-N_2(+O)X_{22}=S_2(+O),\\ \dots\dots\dots\dots\dots\dots\dots\dots\\ M_h(t_h-t_{h-1})X_{h-1,2}-N_h(+O)X_{h2}=S_h(+O),\\ X_{h2}=X_{02},\end{gathered} \tag{27}$$

where

$$M_q(t)U_q=N_q(t), \quad M_q(t)V_q=S_q(t). \tag{28}$$

If $t_1, t_2, \dots, t_h$ were known, one could find all $X_{q2} = X_{q2}(t_1, t_2, \dots, t_h)$ from System (27) and construct all functions $X_q(t)$, that is find* $x_q(t) = \{X_q(t)\}_1$.

* From here on the symbol $\{X_q(t)\}_1$ denotes the first element of the column matrix within the braces.

Then $x(t) = \sum_q xq(t)$ would be a periodic function satisfying (10) and the cross-over Conditions (12). But in this case, the values of $\overline{x}(t)$ at times $t_1, t_2, \ldots, t_h$ might not agree with the given abscissas of the breaks or jumps of the characteristic $\overline{x}(t_q) = \overline{x}_{q\,\text{given}}$ ($q = 1, 2, \ldots, h$) or with the notation System (11)

$$\overline{x}^0_{q2} = \overline{x}_q \text{ given} \tag{29}$$

We now determine $t_1, t_2, \ldots, t_h$ starting from Requirement (29).* Going over from function $\overline{x}(t)$ to $x(t)$ with the help of (7), Condition (29) is written as

$$x^0_{q2} = \overline{x}_{q\,\text{given}} + \rho_{q+1}$$

or

$$\{X_{q2}(t_1, t_2, \ldots, t_h)\}_1 = \overline{x}_{q\,\text{given}} + \rho_{q+1} \quad (q = 1, 2, \ldots, h). \tag{30}$$

Equations (30) serve to determine $t_1, t_2, \ldots, t_h$, and are therefore the required period equations.

Thus to find the unknown times $t_1, t_2, \ldots, t_h$, it is necessary to solve (27) for all X_{q2}, that is to obtain the dependence of each X_{q2} on all unknown passage times of separate portions of the characteristic, and to expect Conditions (30) to be satisfied.

The operation of solving System (27) for the unknowns X_{q2} is reduced to the evaluation of the matrices inverse to M_q and N_q, and to the evaluation of their derivatives. For large $\underline{h}$ and $\underline{n}$ these computations are cumbersome but the problem of construction of period equations is basically reduced to the solution of simultaneous linear algebraic equations and integration of the basic differential equations is not required.

When $h = 2$ (two parts characteristic) it is convenient to solve Equation (27) graphically, and also when $h = 4$ and the behavior is symmetrical. Then the number of unknown times is reduced by half (see [4]), the last Equation (27) is written in the form $X_{h/2,2} = -X_{02}$, and the total number of equations is also reduced by half.

We write Equations (27) showing explicitly all the unknowns x^k_{q2} ($k = 0, 1, \ldots, n-1$; $q = 1, 2, \ldots, h$). Denoting by $H^{(j)}_{kq}(t)$ the elements of matrix $N_q(t)$, analogously to our notation $R^{(j)}_{kq}(t)$ for elements of $M_q(t)$, and by $Q^{(j)}_q(t)$ elements of column matrix $S_q(t)$, we obtain

$$\begin{aligned}
&R_{01}(t_1)\,x^0_{02} + R_{11}(t_1)\,x^1_{02} + \ldots + R_{n-1,1}(t_1)\,x^{n-1}_{02} - \\
&- H_{01}(+O)\,x^0_{12} - H_{11}(+O)\,x^1_{12} - \ldots - H_{n-1,1}(+O)\,x^{n-1}_{12} = Q_1(+O), \\
&R'_{01}(t_1)\,x^0_{02} + R'_{11}(t_1)\,x^1_{02} + \ldots + R'_{n-1,1}(t_1)\,x^{n-1}_{02} - \\
&- H'_{01}(+O)\,x^0_{12} - H'_{11}(+O)\,x^1_{12} - \ldots - H'_{n-1,1}(+O)\,x^{n-1}_{12} = Q'_1(+O), \\
&\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \\
&R^{(n-1)}_{01}(t_1)\,x^0_{02} + R^{(n-1)}_{11}(t_1)\,x^1_{02} + \ldots + R^{n-1}_{n-1,1}(t_1)\,x^{(n-1)}_{02} - \\
&- H^{(n-1)}_{01}(+O)\,x^0_{12} - H^{(n-1)}_{11}(+O)\,x^1_{12} - \ldots - H^{(n-1)}_{n-1,1}(+O)\,x^{n-1}_{12} = Q^{(n-1)}_1(+O), \\
&R_{02}(t_2 - t_1)\,x^0_{12} + R_{12}(t_2 - t_1)\,x^1_{12} + \ldots + R_{n-1,2}(t_2 - t_1)\,x^{n-1}_{12} - \\
&- H_{02}(+O)\,x^0_{22} - H_{12}(+O)\,x^1_{22} - \ldots - H_{n-1,2}(+O)\,x^{n-1}_{22} = Q_2(+O), \\
&\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \\
&R^{(n-1)}_{0h}(t_h - t_{h-1})\,x^0_{h-1,2} + \ldots + R^{(n-1)}_{h-1}(t_h - t_{h-1})\,x^{n-1}_{h-1,2} - \\
&- H^{(n-1)}_{0h}(+O)\,x^0_{02} + \ldots + H^{(n-1)}_{n-1,h}(+O)\,x^{n-1}_{02} = Q^{(n-1)}_h(+O).
\end{aligned} \tag{31}$$

The matrix of System (31) is a matrix of 2n truncated rows; it is of order hn.

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*By this requirement the q-th stage ends just as $\overline{x}(t)$ reaches the value $\overline{x}_{q\,\text{given}}$. In a different case, for example, in a system with backlash, the q-th stage ends with velocity $\dot{\overline{x}}(t)$ becoming zero. For cases of this kind the conditions for ending a successive stage of motion can be generalized, as shown in [1].

When computers are used to calculate the solutions of (31), one can first assume arbitrary values t_q, then solve the algebraical equation to find the unknowns x^k_{q2}, and subsequently to look for such t_q that all $x^0_{q2} = \bar{x}_{q\,\text{given}} + \rho_{q+1}$ ($q = 1, 2, \ldots, h$). This can be done automatically by introducing an aggregate parameter, say $f(t_1, t_2, \ldots t_h) = \sum_{q=1}^{h} (x^0_{q2} - \bar{x}_{q\,\text{given}} - \rho_{q+1})^2 \geqslant 0$, and then by minimizing $f(t_1, t_2, \ldots, t_h)$. If periodic solutions exist, there are also values $t_1, t_2, \ldots, t_h$ making the aggregate parameter equal zero.

Having found values $t_1, t_2, \ldots, t_h$ it should be verified whether they satisfy two conditions: 1) the inequalities $t_1 < t_2 < \ldots < t_h$; should be true; 2) function $x_q(t)$ constructed with the obtained $t_1, t_2, \ldots, t_h$ should at $t = t_q$ reach the value $\bar{x}_{q\,\text{given}} + \rho_{q+1}$ for the first time. The values of $t_1, t_2, \ldots, t_h$, which do not satisfy even one of these conditions should be rejected.

5. Detailed Remarks on Computation

Computation of the derivatives of function $R_{kq}(t)$ may involve difficulties due to the fact that the Fourier series differentiation weakens convergence; divergent series can also occur. To avoid these difficulties we transform (16):

$$R_{kq}(t) = \frac{1}{T} \sum_{r=-\infty}^{\infty} \frac{l^q_k(s)}{L_q(s)} e^{st} =$$

$$= \frac{g^q_{n-k-1}}{g^q_n T} + \frac{1}{T} \sum_{r=-\infty}^{\infty}{}' \frac{l^q_k(s)}{L_q(s)} e^{st} \qquad (s \equiv ir\omega).$$

There $\sum\limits_{r=-\infty}^{\infty}{}'$ denotes the sum with the omitted one term corresponding to $r = 0$.

The easy to verify identity

$$L_q(s) = l^q_k(s)\, s^{k+1} + g^q_n + g^q_{n-1} s + \ldots + g^q_{n-k} s^k,$$

is used to obtain

$$\frac{l^q_k(s)}{L_q(s)} = \frac{1}{s^{k+1}} - \frac{g^q_n}{s^{k+1} L_q(s)} - \frac{g^q_{n-1}}{s^k L_q(s)} - \ldots - \frac{g^q_{n-k}}{s L_q(s)}.$$

Hence

$$R_{kq}(t) = \frac{g^q_{n-k-1}}{g^q_n \cdot T} + \frac{1}{T} \sum_{r=-\infty}^{\infty}{}' \frac{e^{st}}{s^{k+1}} - \frac{g^q_n}{T} \sum_{r=-\infty}^{\infty}{}' \frac{e^{st}}{s^{k+1} L_q(s)} - \ldots$$
$$\ldots - \frac{g^q_{n-k}}{T} \sum_{-\infty}^{\infty}{}' \frac{e^{st}}{s L_q(s)}. \qquad (32)$$

The continuous part $\frac{1}{T} \sum\limits_{r=-\infty}^{\infty}{}' \frac{e^{st}}{s^{k+1}}$ gives a discontinuous saw-tooth function

$$\frac{1}{T} \sum_{r=-\infty}^{\infty}{}' \frac{e^{st}}{s} = \frac{1}{\pi} \sum_{r=1}^{\infty}{}' \frac{\sin r\omega t}{r} = \frac{1}{\pi} \frac{\pi - \omega t}{2} \qquad (0 < t < T).$$

after $\underline{k}$ successive differentiations.

The series cannot be differentiated further, term by term, as it would then become divergent; the function $\frac{1}{\pi}\,\frac{\pi-\omega t}{2}$ can, however, be differentiated directly. Its derivative, equal to $-\frac{\omega}{2\pi}$, is continuous everywhere except at points νT (integral ν), where it is undetermined. The values of the derivative, however, on the left and on the right of point νT are equal; that is at these points the derivative is formally continuous.

The other parts of $R_{kq}(t)$ are continuous together with their derivatives up to the (n −1)th order inclusive. This can be deduced from the form of their Fourier coefficients (see [5]).

We introduce functions

$$\rho_1(t),\ \rho_2(t),\ \ldots,\ \rho_n(t) \tag{33}$$

and

$$\sigma_1^q(t),\ \sigma_2^q(t),\ \ldots,\ \sigma_{2n-1}^q(t), \tag{34}$$

defined as follows:

$$\rho_1(t)=\frac{1}{T}\sum_{r=-\infty}^{\infty}{}'\,\frac{e^{ir\omega t}}{ir\omega}=\frac{1}{\pi}\,\frac{\pi-\omega t}{2},$$

$$\rho_2(t)=\frac{1}{T}\sum_{r=-\infty}^{\infty}{}'\,\frac{e^{ir\omega t}}{(ir\omega)^2}=$$

$$=\int_0^t\rho_1(t)\,dt+\frac{1}{T}\sum_{r=-\infty}^{\infty}{}'\,\frac{1}{(ir\omega)^2}=-\frac{\pi}{6\omega}+\frac{1}{2}\,t-\frac{\omega}{4\pi}\,t^2;$$

$$\rho_3(t)=\frac{1}{T}\sum_{r=-\infty}^{\infty}{}'\,\frac{e^{ir\omega t}}{(ir\omega)^3}=$$

$$=\int_0^t\rho_2(t)\,dt+\frac{1}{T}\sum_{r=-\infty}^{\infty}{}'\,\frac{1}{(ir\omega)^3}=-\frac{\pi}{6\omega}\,t+\frac{1}{4}\,t^2-\frac{\omega}{12\pi}\,t^3$$

$$(0<t<T)$$

and so on;*

$$\sigma_1^q(t)=\frac{1}{T}\sum_{r=-\infty}^{\infty}{}'\,\frac{(ir\omega)^{n-2}}{L_q(ir\omega)}\,e^{ir\omega t},$$

$$\sigma_2^q(t)=\frac{1}{T}\sum_{r=-\infty}^{\infty}{}'\,\frac{(ir\omega)^{n-3}}{L_q(ir\omega)}\,e^{ir\omega t}=\int_0^t\sigma_1^q(t)\,dt+\frac{1}{T}\sum_{r=-\infty}^{\infty}{}'\,\frac{(ir\omega)^{n-3}}{L_q(ir\omega)},$$

$$\sigma_3^q(t)=\frac{1}{T}\sum_{r=-\infty}^{\infty}{}'\,\frac{(ir\omega)^{n-4}}{L_q(ir\omega)}\,e^{ir\omega t}=\int_0^t\sigma_2^q(t)\,dt+\frac{1}{T}\sum_{r=-\infty}^{\infty}{}'\,\frac{(ir\omega)^{n-4}}{L_q(ir\omega)}$$

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* It should be mentioned that for integral k > 0

$$\frac{1}{T}\sum_{r=-\infty}^{\infty}{}'\,\frac{1}{(ir\omega)^{2k-1}}=0,\ \text{and}\ \frac{1}{T}\sum_{r=-\infty}^{\infty}{}'\,\frac{1}{(ir\omega)^{2k}}=\frac{(-1)^k}{\pi\omega^{2k-1}}\,\zeta(2k),$$

where $\zeta(k)$ is Riemann zeta-function tabulated in [6], page 444.

and so on.*

Having now written all functions $R_{kq}^{(j)}(t)$ $(j = 0, 1, \ldots, n-1)$ we note the fact that everyone of them can be represented as a linear combination of Functions (33) and (34) with coefficients $g_n^q, g_{n-1}^q, \ldots\ldots, g_{n-k}^q$. For example

$$R_{kq}^{(n-1)}(t) = -g_n^q \sigma_k^q(t) - g_{n-1}^q \sigma_{k-1}^q(t) - \ldots - g_{n-k}^q \sigma_1^q(t).$$

This fact simplies considerably the computation of the values $R_{kq}^{(j)}(t)$.

Coming now to the problem of computation of $N_q(+O)$ and $S_q(+O)$, we note that due to (28) the question is reduced to finding a method of computation of $M_q(+O)$. When the elements $R_{kq}^{(j)}(t)$ of matrix $M_q(t)$ are constructed by means of Functions (33) and (34) it is obvious that substitution $t = 0$ gives a correct value of matrix $M_q(+O)$. One should especially be careful not to let argument $\underline{t}$ of matrix $M_q(t)$ leave the interval (O, T) as Functions (33) are defined only in that interval.

6. Possible Generalization

Finally, we should note that by using the method given in this paper one is able to reconstruct the periodic solutions of a wider class of simultaneous differential equations, which have on their right-hand side periodic functions $F_i(t)$, period $T = t_h$:

$$\frac{d\bar{x}_i}{dt} = \sum_{j-1}^{n} \alpha_{ij}\bar{x}_j + \varkappa_i y + F_i(t),$$

$$y = f(\bar{x}_1) \qquad (i = 1, 2, \ldots, h).$$

When the functions $F_i(t)$ are smooth and differentiable the required number of times, the derived Equation (2) is of the form (see [1]):

$$D(p)\bar{x} = K(p) f(\bar{x}) + \Phi(t).$$

Equation (10) becomes

$$L_q(p) x(t) = \Phi(t) \quad (t_{q-1} < t < t_q).$$

Simultaneous Equations (26) and (27) for finding the unknowns $X_q(t)$ and X_{q2} becomes as follows:

$$X_q(t) = M_q(t - t_{q-1}) X_{q-1,2} - N_q(t - t_q) X_{q2} - S_q(t - t_q) - P_q(t), \tag{35}$$

$$\begin{aligned} M_1(t_1 - t_0) X_{02} - N_1(+O) X_{12} &= S_1(+O) + P_1(t_1), \\ M_2(t_2 - t_1) X_{12} - N_2(+O) X_{22} &= S_2(+O) + P_2(t_2), \\ \cdots\cdots\cdots\cdots\cdots\cdots \\ M_h(t_h - t_{h-1}) N_{h-1,2} - N_h(+O) X_{h2} &= S_h(+O) + P_h(t_h), \\ X_{h2} &= X_{02}. \end{aligned} \tag{36}$$

On right-hand sides we find now additionally column matrices

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* We note that

$$\sum_{r=-\infty}^{\infty}{}' \frac{(ir\omega)^{n-k}}{L_q(ir\omega)} = 2\sum_{r=1}^{\infty} \mathrm{Re}\, \frac{(ir\omega)^{n-k}}{L_q(ir\omega)},$$

as

$$\frac{(-ir\omega)^{n-k}}{L_q(-ir\omega)} = \overline{\frac{(ir\omega)^{n-k}}{L_q(ir\omega)}}.$$

$$P_q(t) = \begin{Vmatrix} T_q(t) \\ \dot{T}_q(t) \\ \vdots \\ T_q^{(n-1)}(t) \end{Vmatrix}.$$

Function $T_q(t)$ is given by the formula

$$T_q(t) = \frac{1}{T} \sum_{r=-\infty}^{\infty} \mu_{qr} e^{ir\omega t},$$

where

$$\mu_{qr} = \frac{1}{L_q(ir\omega)} \int_{t_{q-1}}^{t_q} \Phi(t) e^{-ir\omega t} dt.$$

As distinct from (26) and (27) the quantity $\omega = 2\pi/t_h$ is known in (35) and (36); there appears, however, a new unknown t_0 which in the self-oscillatory case was equal zero as we were then free to choose the beginning of the time scale.

APPENDIX I

Derivation of Crossing-Over Conditions

In agreement with (7) when $t_{q-1} < t^* < t_q$

$$\bar{x}(t^*) = (t^*) + \rho_q,$$

$$y(t^*) = k_q \bar{x}(t^*) - \lambda_q = k_q x(t^*) + k_q \rho_q + \lambda_q,$$

and when $t_q < t^{**} < t_{q+1}$

$$\bar{x}(t^{**}) = x(t^{**}) + \rho_{q+1},$$

$$y(t^{**}) = k_{q+1} \bar{x}(t^{**}) + \lambda_{q+1} = k_{q+1} x(t^{**}) + k_{q+1} \rho_{q+1} + \lambda_{q+1}.$$

We form the differences

$$\bar{x}(t^{**}) - \bar{x}(t^*) = x(t^{**}) - x(t^*) + \rho_{q+1} - \rho_q$$

and

$$y(t^{**}) - y(t^*) = k_{q+1} x(t^{**}) - k_q x(t^*) + k_{q+1} \rho_{q+1} - k_q \rho_q + \lambda_{q+1} - \lambda_q,$$

and then we make t^{**} and t^* approach t_q from the right and left, respectively. In the (11) notation we obtain

$$\xi_0 = x_{q2}^0 - x_{q1}^0 + \rho_{q+1} - \rho_q,$$

$$\eta_0 = k_{q+1} x_{q2}^0 - k_q x_{q1}^0 + k_{q+1} \rho_{q+1} - k_q \rho_q + \lambda_{q+1} - \lambda_q.$$

Applying the same limit procedure to the differences of functions $\bar{x}(t)$ and $y(t)$ we determine ξ_k and η_k ($k = 1, 2, \ldots, n-1$). Substituting those in (3) we obtain after some rearranging

$$g_0^{q+1} x_{q2}^0 = g_0^q x_{q1}^0 + \psi_{q0},$$

$$g_0^{q+1} x_{q2}^1 + g_1^{q+1} x_{q2}^0 = g_0^q x_{q1}^1 + g_1^q x_{q1}^0 + \psi_{q1},$$

$$\cdots\cdots\cdots\cdots\cdots\cdots$$

$$g_0^{q+1} x_{q2}^{n-1} + \cdots + g_{n-1}^{q+1} x_{q2}^0 = g_0^q x_{q1}^{n-1} + \cdots + g_{n-1}^q x_{q1}^0 + \psi_{q,n-1}.$$

Solving these simultaneous equations for all x_{q1}^{k} ($k = 0, 1, \ldots, n-1$) we shall obtain exactly System (12), where u_{qk} and v_{qk} are determined by the recurrence Formulas (13).

APPENDIX II

Construction of Functions $R_{kq}(t)$

We introduce generalized differential operator p^* (see [4]) which is defined in the following way:

$$p^*F(t) = pF(t) + \sum_q \xi_0^q \delta(t - t_q),$$

$$p^{*2}F(t) = p^2F(t) + \sum_q \xi_0^q \delta'(t - t_q) + \sum_q \xi_1^q \delta(t - t_q),$$

$$p^{*3}F(t) = p^3F(t) + \sum_q \xi_0^q \delta''(t - t_q) + \sum_q \xi_1^q \delta'(t - t_q) + \sum \xi_2^q \delta(t - t_q)$$

etc.

There $p = d/dt$ is the ordinary differential operator; $\xi_0^q, \xi_1^q, \xi_2^q$ etc. are the magnitudes of gaps of the piecewise continuous function $F(t)$ and its derivatives dF/dt, d^2F/dt^2 etc. at time $t = t_q$; $\delta(t)$, $\delta'(t)$ etc. are Dirac impulse function and its derivatives.

For the function $R_{kq}(t)$ according to Condition 2 in Section 3 we have

$$\begin{aligned} R_{kq}(t) &= R_{kq}(t), \\ p^*R_{kq}(t) &= pR_{kq}(t), \\ &\ldots\ldots\ldots \\ p^{*k}R_{kq}(t) &= p^kR_{kq}(t). \end{aligned} \tag{II. 1}$$

In agreement with Condition 3 (Section 3)

$$p^{*(k+1)}R_{kq}(t) = p^{k+1}R_{kq}(t) + \sum_{\nu=-\infty}^{\infty} \delta(t - \nu T). \tag{II. 2}$$

By Condition 4 (Section 3)

$$\begin{aligned} p^{*(k+2)}R_{kq}(t) &= p^{k+2}R_{kq}(t) + \sum_{\nu=-\infty}^{\infty} \delta'(t - \nu T), \\ &\ldots\ldots\ldots\ldots\ldots\ldots \\ p^{*n}R_{kq}(t) &= p^nR_{kq}(t) + \sum_{\nu=-\infty}^{\infty} \delta^{(n-k-1)}(t - \nu T). \end{aligned} \tag{II. 3}$$

Multiplying (II. 1), (II. 2) and (II. 3) by $g_n^q, g_{n-1}^q, \ldots, g_0^q$, the coefficients of operator $L_q(p)$, and adding on the left and right-hand sides we obtain

$$L_q(p^*)R_{kq}(t) = L_q(p)R_{kq}(t) + g_{n-k-1}^q \sum_{\nu=-\infty}^{\infty} \delta(t - \nu T) +$$

$$+ g_{n-k-2}^q \sum_{\nu=-\infty}^{\infty} \delta'(t - \nu T) + \ldots + g_0^q \sum_{\nu=-\infty}^{\infty} \delta^{(n-k-1)}(t - \nu T).$$

If

$$R_{kq}(t) = \sum_{r=-\infty}^{\infty} \gamma_r e^{ir\omega t} \quad \left(\omega = \frac{2\pi}{T}\right),$$

then, using a known formula in Fourier series theory,

$$L_q(p^*) \sum_{r=-\infty}^{\infty} \gamma_r e^{ir\omega t} = \sum_{r=-\infty}^{\infty} L_q(ir\omega)\,\gamma_r e^{ir\omega t},$$

and also from known formulas on Fourier analysis of periodic impulse function $\sum_{\nu=-\infty}^{\infty} \delta(t-\nu T)$ and its derivatives

$$\sum_{\nu=-\infty}^{\infty} \delta(t-\nu T) = \frac{1}{T} \sum_{r=-\infty}^{\infty} e^{ir\omega t}, \qquad \sum_{\nu=-\infty}^{\infty} \delta'(t-\nu T) = \frac{1}{T} \sum_{r=-\infty}^{\infty} ir\omega e^{ir\omega t},$$

$$\sum_{\nu=-\infty}^{\infty} \delta''(t-\nu t) = \frac{1}{T} \sum_{r=-\infty}^{\infty} (ir\omega)^2 e^{ir\omega t} \text{ etc.}$$

and by taking into account Condition 1 we obtain

$$L_q(ir\omega)\,\gamma_r = \frac{1}{T}\left[g^q_{n-k-1} + g^q_{n-k-2}\, ir\omega + \ldots + g^q_0 (ir\omega)^{n-k-1}\right],$$

hence

$$\gamma_r = \frac{1}{T}\,\frac{l^q_k(s)}{L_q(s)} \quad (s \equiv ir\omega).$$

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SUMMARY

The paper deals with determination of periodic behavior of control systems having an arbitrary piece-linear characteristic. The said periodic behavior is determined as a complete Fourier series without neglecting harmonics. The problem is reduced to solving simultaneous transcendental equations that determine motion time in each part of the nonlinear characteristic. Some problems of using computers to solve period equations are considered.

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CONCERNING THE EQUIVALENCE OF PULSE AND CONTINUOUS-DATA CONTROL SYSTEMS

V. A. Rubtsov

Abstract

The insufficiency of the generally used equivalence criterion of pulse and continuous-data control systems is established. Sufficient conditions for the equivalence of closed loop control systems are determined.

It is usually assumed that the analysis of a pulse control system can be reduced to the analysis of the corresponding continuous-data system, if the repetition period T_r (equal to the time interval between two consecutive error measurements) is small compared to the basic (essential) time constants of the linear part of the pulse control system [1-4]. However, it will be subsequently shown that considerable differences can exist between the closed loop, pulse and continuous control systems under comparison, both with respect to transient processes as well as stability conditions.

Derivation of Sufficient Conditions for the Equivalence of Linear, Closed Loop, Pulse and Continuous-data Control Systems

Let us examine a series of preliminary proposals which will permit us to determine the sufficient conditions for the equivalence of the pulse and continuous control systems being compared.

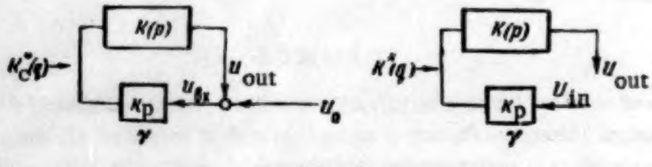

Fig. 1.

1. Sufficient conditions for the equivalence of closed loop, pulse and continuous control systems cannot be established from the comparison of the transfer functions of the open loop control systems.

A linear, pulse control system usually includes its own linear part, with transfer function K(p), and a pulse feedback loop with the amplification factor of the pulse unit K_p and the parameter γ.

Blocks schematics of the closed, and open loop, pulse control system are shown in Fig. 1, where the following designations are introduced: $K_c^*(q)$ — transfer function of the closed loop control system, $K^*(q)$ — transfer function of the open loop control system, γ — a parameter characterizing the mark-space ratio of the error measurements (of the succession of pulses), $0 < \gamma \leq 1$, $K(p) = \frac{P(p)}{Q(p)}$, P(p) and Q(p) are polynomials in p. T_r will be used to designate the pulse repetition period.

Let us express the transfer function of the linear part K(p) in the form of a fraction-rational function, making, at the same time, the substitution $p = q/T_r$:

$$K(q) = k_0 \frac{B_m q^m + B_{m-1} q^{m-1} + \ldots + b_1 q + b_0}{a_n q^n + a_{n-1} q^{n-1} + \ldots + a_1 q + a_0}. \tag{1}$$

Here $a_n = 1$, a_i are functions of the systems time constants T_i and repetition period T_r, k_0 is a constant coefficient (the amplification factor of the linear part).

Expression (1) can also be expressed in the form

$$K(q) = k_0 \sum_{\nu=1}^{n} \frac{P(q_\nu)}{Q'(q_\nu)} \frac{1}{q - q_\nu}, \tag{1'}$$

where $q_\nu = T_r p_\nu$, p_ν – are simple poles of the transfer function of the linear part of the control system with a pulse feedback loop.

If the condition

$$\lim_{q_\nu \to \infty} K(q_\nu) = 0, \tag{2}$$

holds, i.e., in Expression (1) the degree of the numerator m is less than the degree of the denominator n by at least one, then the transfer function of the open loop control system, including the pulse unit, can be written in the form

$$K^*(q) = k_0 k_p \sum_{\nu=1}^{n} \frac{P(q_\nu)}{Q'(q_\nu)} \frac{1 - e^{-q_\nu \gamma}}{q_\nu} \frac{e^{q_\nu}}{e^q - e^{q_\nu}}. \tag{3}$$

The mathematical expression for a practical criterion which makes it possible to consider a pulse control system as an ordinary system responding to a continuously measured error, is usually written in the form [2]

$$T_n p_\nu| = |q_\nu| \ll 1. \tag{4}$$

The physical significance of Inequality (4), as it is well known, is that all, or at least the basic (essential) time constants of the linear part of a control system are large as compared with the repetition period T_r [2].

Moreover, Expression (4) emerges as the result of the establishment of equivalence between the transfer function of an open loop, continuous-data control system (1) and the transfer function of an open loop pulse control system (3), for the limiting transition, i.e., for $T_r \to 0$.

In fact, Condition (4) is necessary and sufficient for the establishment of equivalence between pulse circuits and corresponding continuous-data circuits, if one considers the usual pulse circuits which do not contain a feedback loop. Because the poles of the transfer function of the open loop, pulse control system coincide with the poles of the transfer function of the linear part of the system, or differ from them by $\pm 2\pi kj$, i.e.,

$$\overset{*}{q}_\nu = q_\nu \quad \text{or} \quad \overset{*}{q}_\nu = q_\nu \pm 2\pi kj, \tag{5}$$

then in the region $\pi < \operatorname{Im} q_\nu \leq \pi$ the minor poles q_ν^* will always correspond to the minor poles q_ν (when $|q_\nu| \ll 1$). Consequently, for $T_r \to 0$, when $|q_\nu| \ll 1$), it is permissible to make, in Expression (3), not only the substitution $e^{q_\nu} = 1 + q_\nu$, $e^{q_\nu \gamma} = 1 + \gamma q_\nu$, but also $e^q = 1 + q$. It is not hard to see that, after this substitution in Expression (3), the transformer function of the open loop pulse control system will differ from the transfer function of the linear part of this system only by a constant multiplier $k_p \gamma$. However, the insufficiency of Criterion (4) for closed loop pulse control systems becomes obvious if one examines a few concrete examples.

Let us examine the simplest static control system using a nonperiodic element with a time constant T.

Let $K(q) = k_0 \frac{\beta}{q+\beta}$, where $\beta = T_r/T$ and $\gamma = 1$. The behavior of the closed loop control system is determined by its transfer function and, partially, by the poles of this function. Making use of (3), the pole of the transfer function of the closed loop control system can be expressed in the form

$$q_c^* = -\beta + \ln[e^{-\beta} - A(1 - e^{-\beta})], \tag{6}$$

where $A = k_0 k_p \gamma$. Correspondingly, the pole of the transfer function of the continuous-data system will be

$$q_c = -\beta(1 + A). \tag{7}$$

Obviously, the equivalence of the systems can only be considered when

$$q_c^* \approx q_c. \tag{8}$$

Let us suppose that Condition (4) is satisfied, in which case (6) can be written in the form:

$$q_c^* \approx \ln[1 - \beta(1 + A)]. \tag{6'}$$

If we follow the commonly accepted approach [2-4], then Condition (8) must always be satisfied if Inequality (4) holds. However, as can readily be seen from (6') and (7), this will only be true when $A\beta << 1$, i.e., for a sufficiently small feedback factor A (small amplification in the open loop pulse control system).

From (6') and (7) it follows immediately that, for $2 > A\beta \geq 1$, the systems differ considerably, even when Condition (4) is satisfied. Thus, for example, in the continuous-data system, the transient process will be nonperiodic for any limiting values of $A\beta$, while in the pulse control system, for $2 > A\beta \geq 1$, the transient process will be oscillatory. Moreover, the pulse control system will already be unable to control when $A\beta \approx 2$, because it will be self-exciting.

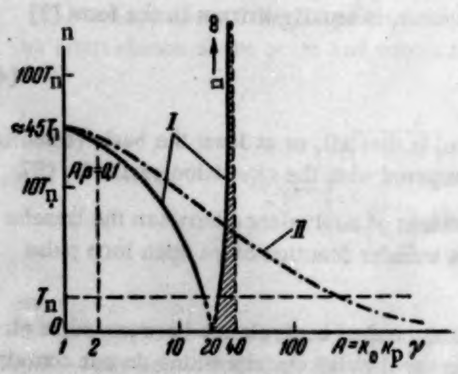

Fig. 2.

What has been said is illustrated in Fig. 2, which shows a curve of the function $n = f(A)$, when $\underline{n}$ is the duration of the transient process, expressed in repetition periods T_r, I is the curve showing the behavior of the pulse control system when $\beta = 0.05$ and $\gamma = 1$, II is the curve showing the behavior of the compared* continuous-data system. The region in which the transient process in the pulse control system is of an oscillatory nature is shown shaded in Fig. 2. In determining the duration of the transient process $\underline{n}$ it was assumed that the transient process ends when $u_{out\,\infty} - u_{out} = \pm 0.1\, u_{out\,\infty}$, where $u_{out\,\infty}$ is the stable value.

Thus, from the examination of the above example of the simplest form of control system it becomes apparent that Condition (4), which results from the analysis of compared open loop, pulse and continuous-data control systems, is not sufficient for the equivalence of these systems in the closed loop condition. Inequality (4) is, in the general case, a necessary but not sufficient condition for the equivalence of closed loop automatic control systems.

In the examination of closed loop, pulse control systems, the behavior of the system is characterized by the transfer function of the closed loop system, which can be written in the form:

$$K_c^*(q = \frac{K^*(q)}{1 + K^*(q)}, \tag{9}$$

* By a compared continuous-data system is understood such a continuous control system into which a pulse control system degenerates for the limiting value of the feedback factor, when $T_r \to 0$.

where $K^*(q)$ is the transfer function of the open loop system, determined by Expression (3).

For the compared continuous control system, the transfer function can be expressed in the form:

$$K_c(q) = \frac{K(q)}{1 + k_p \gamma K(q)} . \tag{10}$$

Obviously, the compared closed loop systems will be equivalent under the condition that the denominator of Expression (9) will degenerate into the denominator of Expression (10). From the formal identity of expressions $K^*(q)$ and $K^*_c(q)$ as well as $K(q)$ and $K_c(q)$, it is obvious that the conditions for the equivalence of the closed loop, pulse and continuous-data control systems will be analogous to the conditions for the equivalence of the open loop systems.

In establishing the equivalence of the open loop, pulse and continuous-data control systems it was assumed that it was necessary to fulfill Conditions (4) and (5).

Therefore, for the equivalence of the closed loop, control systems, two analogous conditions must be satisfied, namely:

$$q^*_{c\nu} = q_{c\nu}, \quad |q^*_{c\nu}| \ll 1. \tag{11}$$

Conditions (11) are the determining ones in the establishment of a practical criterion for the equivalence of closed loop, pulse and continuous-data control systems.

In fulfilling (4), the denominator in (9) can be expressed in the form

$$\begin{aligned} G^*_c(q^*_c) = \prod_{\nu=1}^{n} Q'(q_\nu)\left(e^{q^*_c} - 1 - q_\nu\right) + \\ + A\left\{\sum_{\nu=1}^{n} \frac{P(q_\nu)}{Q'(q_\nu)} \frac{1}{e^{q^*_c} - 1 - q_\nu} \prod_{\nu=1}^{n} Q'(q_\nu)\left(e^{q^*_c} - 1 - q_\nu\right)\right\} \end{aligned} \tag{12}$$

and, correspondingly, the denominator of (10) can be written in the form

$$G(q_c) = \prod_{\nu=1}^{n} Q'(q_\nu)(q_c - q_\nu) + A\left\{\sum_{\nu=1}^{n} \frac{P(q_\lambda)}{Q'(q_\nu)} \frac{1}{q_c - q_\nu} \prod_{\nu=1}^{n} Q'(q_\nu)(q_c - q_\nu)\right\} . \tag{13}$$

If one assumes that the second Condition in (11) is realized, then, obviously, it is permissible to make, in Expression (12), the substitution* $e^{q^*_c} = 1 + q^*_c$. After this substitution, expressions $G^*_c(q^*_c)$ and $G_c(q_c)$ coincide completely, and the behavior of the pulse control system will not differ from the behavior of the continuous-data system; in other words, the systems will be equivalent.

In this way, Conditions (11) will be necessary and sufficient conditions for the equivalence of closed loop, pulse and continuous-data control systems. However, Conditions (11) are of little practical value since they require the determination of the numerical values of the poles of the closed loop, pulse control system's transfer function.

In practice, it is desirable that the sufficient conditions for the equivalence of pulse and continuous-data control systems be completely determined by the characteristics of the continuous control system itself, and do not require the direct analysis of the pulse control system. It should be noted that, for closed loop control systems, Criterion (4) would be sufficient only in the case where, from the condition $|q_\nu| \ll 1$, condition $|q^*_{c\nu}| \ll 1$ would always follow. However, such a mutual relationship does not occur in general.

*In paper [1], in establishing the equivalence of pulse and continuous control systems, it was postulated that the expansion $e^{q_c} = 1 + q_c$ was permissible. However, such a substitution is far from always possible.

2. The minor roots of the equation $G_c(q_c) = 0$ correspond to the minor roots of equation $G_c^*(q_c^*) = 0$.

In the case where the linear portion of the pulse control system can be expressed as a fraction – rational function of the form (1), whenever Inequality (4) is satisfied, there will always be a definite relationship between the roots of the equations $G_c(q_c) = 0$ and $G_c^*(q_c^*) = 0$, i.e., between the poles of the transfer functions of the closed loop pulse and continuous-data control systems. This relationship can be expressed in the form

$$q_c^* \approx \ln(1 + q_c). \tag{14}$$

Expression (14) results from the fact that $G_c^*(q_c^*)$ can be formed directly from $G_c(q_c)$, whenever (4) is satisfied, by means of the substitution $q_c = z - 1$ [2] in $G_c(q_c)$, where $z = e^{q_c^*}$. Expression (12) was obtained exactly in this way.

If, in (14), $|q_{c\nu}| \ll 1$ then $|q_{c\nu}^*|$ will also be quite small and the approximate equality of the poles results, which can be written in the form

$$q_{c\nu}^* = (1 \pm \delta)\, q_{c\nu} \tag{15}$$

where $\delta \ll 1$.

From what has been said above it follows that it is sufficient to determine the condition for the transfer function of the closed-loop, continuous-data system to have small poles, to thereby establish the required sufficient conditions for the equivalence of the compared pulse and continuous-data control systems.

3. For certain values of the coefficients in the denominator of Expression (1), the roots of the denominator will have small values, i.e., $|q_\nu| \ll 1$.

The coefficients of the denominator of transfer function (1) are related to its roots thru Viet's equations. This relationship can be expressed by means of the following equalities:

$$\begin{aligned} a_n &= 1, \\ a_{n-1} &= -(q_1 + q_2 + \cdots + q_n), \\ a_{n-2} &= (q_1 q_2 + q_1 q_3 + \cdots + q_{n-1} q_n), \\ a_{n-3} &= -(q_1 q_2 q_3 + q_1 q_2 q_4 + q_1 q_3 q_4 + \cdots + q_{n-2} q_{n-1} q_n, \\ &\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots \\ a_0 &= (-1)^n q_1 q_2 \cdots q_n. \end{aligned} \tag{16}$$

From (16) the following inequalities follow:

$$\begin{aligned} a_{n-1} &\leqslant n\,(q_i)_{\max} \\ a_{n-2} &\leqslant (n-1)\,(q_i q_j)_{\max} \\ a_{n-3} &\leqslant (n-2)\,(q_i q_j q_k)_{\max} \\ &\cdots\cdots\cdots \\ a_0 &\leqslant q_{i\,\max}^n \end{aligned} \tag{17}$$

where $q_{i\,\max}$ is the largest root, $(q_i q_j)_{\max}$ is the largest product of a pair of roots, $(q_i q_j q_k)_{\max}$ is the largest product of three roots, etc. We will conditionally assume those roots to be small for which the following relationship holds

$$|q_\nu| \leqslant \sigma, \tag{18}$$

where $\sigma \ll 1$.

If (18) is fulfilled then Inequalities (17) can be replaced by the following conditions:

$$\begin{aligned} a_{n-1} &\leqslant n\sigma, \\ a_{n-2} &\leqslant (n-1)\,\sigma^2, \\ a_{n-3} &\leqslant (n-2)\,\sigma^3, \\ &\cdots\cdots\cdots \\ a_0 &\leqslant \sigma^n. \end{aligned} \tag{19}$$

From (19) it follows that if the roots of the denominator of Expression (1) are small, its constant coefficients are also small or, at least, cannot be greater than the values determined by Inequalities (19).

Let us limit the class of polynomials expressing the denominator of transfer function (1), to those polynomials which satisfy the Hurwitz conditions. In that case it turns out that, if Conditions (19) are satisfied, the roots of the denominator of (1) will always be small compared to unity and on the other hand, if the values of the coefficients do not satisfy these inequalities, then the conditions for smallness of the roots are not fulfilled, at least for one or several roots.

4. Sufficient conditions for the equivalence of closed loop, pulse and continuous-data control systems are determined not only by the systems time constants T_1, but also by the feedback factors A.

Taking (10) into account, let us write the denominator of the transfer function of the continuous-data system being compared in the form

$$G_c(q_c) = 1 + k_p\gamma K(q_c). \tag{20}$$

In accordance with (1), for $a_n = 1$, the right side of (20) can be expressed in the form

$$\begin{aligned} 1 + k_c\gamma K(q_c) = q_c^n + (a_{n-1} + Ab_{n-1})\, q_c^{n-1} + \\ + (a_{n-2} + Ab_{n-2})\, q_c^{n-2} + \cdots + (a_1 + Ab_1)\, q_c + a_0 + Ab_0. \end{aligned} \tag{20'}$$

It follows from (20') that the coefficients of the denominator of the transfer function of a closed loop control system are functions of not only the time constants of the different loops in the system and the repetition period T, but also of the parameter A. In present pulse control systems, the value of A can reach thousands and even tens of thousands and, at the same time, the coefficients of Equation (20') can not be small, even for sufficiently large time constants of the system, when Criterion (4) is fulfillable.

In the investigation of a closed loop control system, Conditions (19) will be satisfied, in general, only for certain values of parameter A. In accordance with (19), these values of A will be determined by the following inequalities:

$$\begin{aligned} a_{n-1}\left(1 + A\frac{b_{n-1}}{a_{n-1}}\right) &\leqslant n\sigma, \\ a_{n-2}\left(1 + A\frac{b_{n-2}}{a_{n-2}}\right) &\leqslant (n-1)\,\sigma^2, \\ a_{n-3}\left(1 + A\frac{b_{n-3}}{a_{n-3}}\right) &\leqslant (n-2)\,\sigma^3, \\ &\cdots\cdots\cdots \\ a_0\left(1 + A\frac{b_0}{a_0}\right) &\leqslant \sigma^n, \end{aligned} \tag{21}$$

where $\sigma \ll 1$.

The roots of the denominator of the transfer function of the closed loop, continuous-data system (10) will be very small compared to unity specifically for those values of A which satisfy Inequalities (21). Consequently, to the extent that (15) will be satisfied at the same time, Conditions (21) will be the sufficient conditions for the equivalence of the closed loop, pulse and continuous data systems.

The degree of the inequality $\sigma \ll 1$ determines the accuracy of the coincidence of the basic characteristics of the transient processes and the stability conditions, and the greater this inequality, the more are the compared systems equivalent. For most practical cases it is sufficient to take $\sigma = 0.1$. Thus, for example, in the example investigated above, when $\sigma = 0.1$, the difference between the durations of the transient process in the compared systems will not exceed 1%.

It should be noted that complete equivalence of the systems, i.e., their equivalence not only with respect to the nature of the transient processes, but also with respect to stability, requires that the sufficient conditions be fulfilled even for the value of A corresponding to the excitation condition, i.e., for $A = A_{\infty}$. However, the compared systems can be equivalent with respect to transient processes by fulfilling sufficient conditions not depending on the fulfillment of the conditions for total equivalence. An example of such a system is a continuous-data control system, described by a third order differential equation, for which, in Expression (1), the coefficients $b_{n-1} = b_{n-2} = 0$ and $b_0 = a_0$.

In fact, if the linear portion of a control system, having a pulse feedback loop, represents a three-section filter with equal time constants, then the feedback factor A (amplification factor) for which the system becomes excited will be equal to 8 [5] irrespective of the absolute value of the time constants. Hence, by increasing the system's time constants one can always satisfy sufficient Conditions (21) for $A = A_{\infty}$, i.e., to have the compared systems completely equivalent.

However, if one of the coefficients b_{n-1} or b_{n-2} is not equal to zero, then the continuous-data system being compared can, as it is well known, remain stable for any large value of A [5]. Consequently, in this case, the condition for complete equivalence will not hold and the conditions for the stability of the pulse control system can be determined only by the direct analysis of such a system.

SUMMARY

1. The condition for the equivalence of pulse and continuous-data control systems

$$|q_\nu| = |T_r p_\nu| \ll 1$$

is necessary but not sufficient in the investigation of closed loop control systems.

2. To establish the equivalence of the systems with respect to transient processes it is necessary that, besides Condition (4), sufficient Conditions (21) be satisfied.

3. The investigation of the stability of closed loop, pulse control systems, based on the analysis of the corresponding continuous-data systems is, in general, not possible. Such an investigation is possible for a particular class of closed-loop systems for which sufficient Conditions (21) are satisfied even for $A = A_{\infty}$.

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CONCERNING THE EXISTENCE OF A CYCLE BEYOND THE ABSOLUTE STABILITY CONDITIONS OF A THREE DIMENSIONAL SYSTEM*

B. V. Shirokorad

Abstract

Beyond the absolute stability conditions (according to Lur'e-Letov [1]) of a closed-loop system with a neutral object and a regulator having a nonlinear speed of control reset, there always exists a nontrivial (in particular, periodic) stable regime. Physical interpretations of the system are presented.

A study is made of the properties of the phase portrait of the simplest type of automatic control system [2] described by the equations

$$\begin{aligned}\dot{x}_1 &= -\rho_1 x_1 + f(\sigma),\\ \dot{x}_2 &= -\rho_2 x_2 + f(\sigma),\\ \dot{\sigma} &= \beta_1 x_1 + \beta_2 x_2 - f(\sigma),\end{aligned} \qquad (1)$$

for the critical case of a neutral object $\rho_1 = 0$ when $\beta_1 = \beta_2 > \rho_2 > 0$.

If, in a finite interval, the function $f(\sigma)$ belongs to A** class, if it is a Lipshits-type and is nonconvex (part 2), it satisfies the conditions of dissipativity (20), repulsivity (26), and Bellman's condition (27)***, then there exists a sphere S and a parallelipiped P situated inside it which contains a closed-loop trajectory-cycle (stable regime, part 6). Trajectories which get into P at t = 0 are contained in S when t > 0 (are stable according to Lagrange) and return into P when t > T > 0 (mainly dissipative, according to Massera, part 2). Any trajectory passing thru a point in P and not tending toward the origin as $t \to +\infty$, has among its ω-defined set [3, 4] an oscillatory regime (part 4). In the vicinity of the origin there exist only two unstable, in the Liapunov sense, trajectories (part 3), tending to the origin as $t \to +\infty$.

In the analysis of automatic control systems, in particular (1), the basic problem is the determination of the steady state, stable regimes (part 6) and the investigation of transient processes characterizing the method by which other (transient) regimes approach these stable ones.

In the conditions for total asymptotic stability $\beta_2 < \rho_2 \int_0^\sigma f(x)\,dx \to \infty$, when $|\sigma| \to \infty$ [2], the phase portrait of System (1), in the case of a neutral object, does not contain any closed trajectories (nontrivial, stable modes). All the other (transient) regimes, which fill the portrait, approach the origin (zero solution or trivial, stable regime). For $\beta_2 > \rho_2$ there always exists in the A class such an $f(\sigma)$, for which the portrait contains at least one trajectory which leaves a sufficiently small region around the origin for a sufficiently large t > 0 (part

* Delivered on October 9, 1957 at I.A.T., AN SSSR during the seminar on automatic control.

** To the A class belong certain, piecewise-continuous, with a finite number of discontinuities of the 1st type, apriori fixed functions $f(\sigma)$, satisfying the condition $f(\sigma) > 0$ when $|\sigma| > \sigma_* \geq 0$ and $f(\sigma) = 0$ when $|\sigma| \leq \sigma_*$. For the sake of simplicity, henceforth $\sigma_* = 0$.

*** For example, if the function $f(\sigma)$ is piecewise-linear.

3). At the same time, it might turn out apriori that the portrait contains some finite region, containing the origin, from which the inside trajectories do not emerge for $t > 0$, and into which enter all the outside ones for sufficiently large $t > 0$. If the diameter of this region is so small that the technical requirements on the accuracy of control are satisfied, then recommendations concerning the adjustment of the control unit, based on the theory of absolute stability, can become incorrect.

These and many other factors make it interesting to study not only the local properties of the zero solution of (1), but also the properties of the phase portrait as a whole.

The framework of the problems of B. V. Bulgakov [5-7] is part of the work started back in the '40's by A. A. Andronov [8, 9]. It complements the work of A. I. Lur'e [1] and A. M. Letov [2, 10] and, on the other hand, generalizes the work of Giuseppe Colombo [11]* for the case when the coefficient of coupling between the grid and plate circuits α vanishes (see § 2, (6), [11]), which corresponds, in our case, to Condition (7) (neutrality of the object $\rho_1 = 0$, $\rho_2 > 0$, see further on), while the function $f(\sigma)$ belongs to the A class, where neither its smoothness nor its symmetry are presumed.

In System (1) being studied belongs to a general class of controlled systems, examined in [13], to which, in particular, belong the systems investigated in [14]. The simple apparatus used for the study of the phase portrait of a three-dimensional system in [11] and in this work, can be applied to other types of equations, to those studied for example in [15], or to systems containing several nonlinear functions.

Probably the statement of the problem dealt with in this work, concerning the investigation of the phase portrait of (1) beyond the necessary and sufficient (if one excludes from the investigation the bifurcational case of the boundary) conditions for absolute stability [16]:**

$$p_1 + \mathrm{P} > 0, \quad p_2 + \Sigma > 0, \quad \sqrt{p_1 + \Pi} < \sqrt{\mathrm{P}} + \sqrt{\Sigma(p_2 + \Sigma)},$$

where

$$p_1 = -(\rho_1\beta_2 + \rho_2\beta_1), \quad p_2 = -(\beta_1 + \beta_2), \quad \mathrm{P} = \rho_1\rho_2, \quad \Sigma = \rho_1 + \rho_2.$$

is original.

1. Physical Interpretations

Let us consider in our analytical investigation two physical interpretations of System(1). The first is an automatic control system [2], consisting of an aeroplane (Fig. 1) and a stabilizer (autopilot) controlling the bank angle γ (rotation about its axis of symmetry $\overline{OX}_1$). The second is the classical circuit of a single tube oscillator (Fig. 2) [11].

Turning to Fig. 1, let us consider the simplest system for the automatic piloting of a'plane, flying along a straight course, at a constant velocity, at a given height above the surface of the earth. We will assume that the channel for stabilization with respect to the $\overline{OX}_1$ axis depends so little on the two other channels (with respect to the $\overline{OY}_1$ and $\overline{OZ}_1$ axes) that the absolute angular velocity $\omega_x = \gamma$ of the 'plane's rotation about axis $\overline{OX}_1$ depends only on the deflection δ_a of the aileron (bank control) from its neutral position.

*The basic theoretical principles of [11] and the present article can be found in the popular report of S. A. Stebakov [12]. The author of [11] admitted a series of inaccuracies into his paper and even an incorrect evaluation of the value of $\underline{y}$ (see (24') and above, in paper [11]).

**In a two-dimensional, speed and position control plane, the quantities p_1 and p_2 take on a very definite physical sense: p_1 is the position transfer ratio of the stabilizer, p_2 is the velocity transfer ratio.

The proof of these conditions [16] required that $\mathrm{P} > 0$; however, this limitation is not essential since, in our case, when $\rho_1 = 0$, $\rho_2 > 0$, they coincide with the previously known $\beta_1 < 0$, $\beta_2 < \rho_2$ [2], which guarantee sufficiency. The necessity is proven by the function of the A class $f(\sigma) = D\sigma$, D = const, satisfying the repulsivity Condition (26) and upsetting the stability when $\beta_2 > \rho_2$.

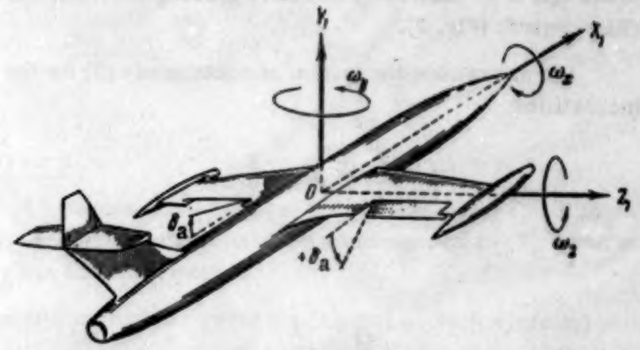

Fig. 1.

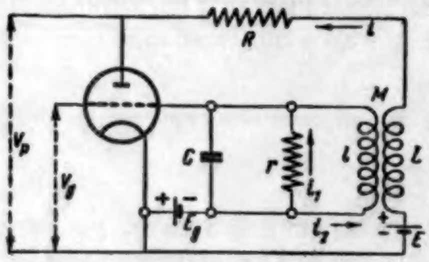

Fig. 2.

Introducing into the analysis, in our case, the constant, negative, aerodynamic coefficients $M_x^{\omega_x}$ kg sec^2 of natural damping, and $M_x^{\delta_a}$ kg of the aileron's effectiveness, we can describe the banking motion by the equation

$$I_x\dot{\omega}_x = M_x^{\omega_x}\omega_x + M_x^{\delta_a}\delta_a, \tag{2}$$

where I_x kg sec^2 is the constant moment of inertia of the aeroplane about its principal axis $\overline{OX_1}$. The bank autopilot, with strong feedback coupling, is described by the equations

$$\dot{\delta}_a = f(\sigma), \quad \sigma = k_\gamma\gamma - \delta_a, \tag{3}$$

where the positive, constant quantity k_γ is called the autopilot's transfer ratio, while the class A function $f(\sigma)$ is the speed characteristic of the steering machine.

The system of Equations (2) and (3) are reduced to the form of (1) by the following substitution:

$$\begin{gathered}
\rho_1 = 0, \quad \rho_2 = \frac{-M_x^{\omega_x}}{\sqrt{-M_x^{\delta_a}I_x}}, \quad -\beta_1 = \beta_2 = \frac{k_\gamma}{\rho_2}, \\
\delta_a = x_1, \quad \omega_x = \frac{-M_x^{\omega_x}}{I_x}(x_2 - x_1), \quad \tau = t\sqrt{\frac{-M_x^{\delta_a}}{I_x}},
\end{gathered} \tag{4}$$

where τ is a dimensionless time parameter.

Turning now to the circuit of the single tube oscillator (Fig. 2), we will use i, V_p and V_g to designate the triode's plate current, plate and grid potentials, respectively. If μ is the amplification factor, then $i = \varphi(V_p + \mu V_g)$, where the continuous function φ (of the A class) is called the triode characteristic (Fig. 3).

Furthermore, let E and E_g represent, respectively, the sources of emf in the plate and grid circuits, C, l, and r the capacitance, inductance and resistance in the grid circuit; L and R the inductance and resistance in the plate circuit; i_1 and i_2 the currents in the grid circuit, passing thru the resistance r and inductance l.

If M is the mutual inductance between the plate and grid circuits (feedback factor) then the oscillator equations take on the following form:

$$\begin{aligned}
&V_g = E_g - l\dot{i}_2 - M\dot{i}, \\
&\dot{V}_g C = i_1 + i_2, \\
&V_p = E - Mi_2 - L\dot{i} - Ri, \\
&g(i) = V_p + \mu V_g,
\end{aligned} \tag{5}$$

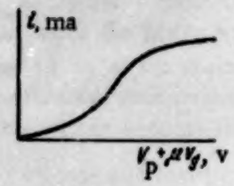

Fig. 3.

where g(i) is an essentially smoothly growing function, namely, the triode's reverse characteristic (Fig. 3).

Let us examine the system of equations in (5) for the particular case when the inequalities

$$\sqrt{\frac{L}{l}} < \mu < \frac{E}{E_g} \tag{6}$$

and the equality

$$M = \frac{L}{\mu}, \tag{7}$$

are satisfied, which is always possible by proper selection of parameters.

To reduce (5) to the normal Cauchy form of three equations, we will introduce in Conditions (6) and (7), the designations

$$\rho_2 = \frac{1}{Cr}, \quad k_\gamma = \frac{1}{Cl} \tag{8}$$

and examine the equation in i

$$Ri + g(i) = E - \mu E_g,$$

which has the single solution $i = i_0 > 0$. This solution determines the steady state value (stable or unstable) of the triode's plate current.

Let us further designate the deviation of the plate current i from the quiescent i_0 by means of

$$x = i - i_0 \tag{9}$$

and introduce into the analysis the function

$$f(x) = \frac{l}{M(\mu l - M)}\left[Rx + g(i_0 + x) - g(i_0)\right], \tag{10}$$

which vanishes together with the deviation x and grows smoothly, since it follows from (6) that

$$\mu l - M = \frac{l}{\mu}\left(\mu^2 - \frac{L}{l}\right) > 0.$$

If we now introduce into the analysis the new variables

$$y = \dot{x} + f(x), \quad z = \dot{y}, \tag{11}$$

then, taking (8-11) into account, System (5) can be reduced to the normal form

$$\begin{aligned} \dot{x} &= y - f(x), \\ \dot{y} &= z, \\ \dot{z} &= -\rho_2 z - k_\gamma f(x), \end{aligned} \tag{12}$$

to which System (1) also reduces by means of the substitution

$$\begin{aligned} x &= \sigma, \\ y &= \beta_1 x_1 + \beta_2 x_2, \\ z &= -k_\gamma x_2, \end{aligned} \tag{13}$$

only if*

$$\rho_1 = 0, \quad \rho_2 > 0, \quad -\beta_1 = \beta_2 = \frac{k_\gamma}{\rho_2} \tag{14}$$

2. Lagrange Stability

For a Lipshits-type,** nonconvex*** $f(x)$ of the A class, under Conditions (14), is examined in Systems (1), expressed in the form (12) belonging to a class of dynamic equations [17], because the right sides of its equations do not contain explicitly the time parameter t.

For the sake of brevity, we will designate the integral curve (trajectory) passing thru the point $X(0) = \{x(0), y(0), z(0)\}$ at $t = 0$, by $X(t) = X[t, X(0)]$, where $X(t) = \{x(t), y(t), z(t)\}$, $(X(0) = X[0, X(0)])$ is a vector describing the point's motion.

If there exists a sphere S, with radius R, within which is contained the whole trajectory $X(t)$, i.e., $|X(t)| < R(|X(t)| \equiv (x^2 + y^2 + z^2)^{1/2})$ for every t, then the trajectory $X(t)$ is called stable according to Lagrange; if this condition only holds for $t > 0$ $(t < 0)$, then $X(t)$ is called stable according to Lagrange in the positive (negative) direction.

Let us prove that there exists such an open parallellipiped P

$$-r \leqslant \underline{\xi} < x < \xi \leqslant r, \quad |y| < r, \quad |z| < r \tag{15}$$

(where $\underline{\xi}$ and ξ are some quantities) that every trajectory $X(t)$ of System(12) which passes through the point $X(0) \in P$ at $t = 0$, does not emerge from the sphere $S(X(t) \in S)$ when $t > 0$, i.e., it is stable according to Lagrange in the positive sense. Moreover, there exists a number $T > 0$ such that $X(t) \in P$ when $t > T$, or System (12) is mainly dissipative (belongs to Massera's D-class [18]). At the same time each trajectory $X(t) = X[t, X(0)]$ $(X(0) \in P)$ is determined for an infinite time interval, since afortiori the following theorem [3] takes effect: if the trajectory of motion remains, with increase in time, in a closed, bounded region, then the motion can be extended over an infinite interval $[t_0, +\infty]$.

For the proof let us examine the system

$$\ddot{y} + \rho_2 \dot{y} + k_\gamma y = k_\gamma \dot{x}, \quad \dot{x} = y - f(x), \tag{16}$$

which is obtained from (12) if z is excluded.

Let us suppose apriori, that the function $x = x(t)$ is known and the conditions of absolute stability will be violated, i.e., $\beta_2 = \frac{k_\gamma}{\rho_2} > \rho_2$.

Then

$$y = \left(C_1 \cos\frac{\lambda}{2} t + C_2 \sin\frac{\lambda}{2} t\right) \exp\left(-\frac{\rho_2}{2} t\right) + \\ + \frac{2k_\gamma}{\lambda} \int_0^t \dot{x}(\tau) \sin\frac{\lambda}{2}(t - \tau) \exp\frac{\rho_2}{2}(\tau - t)\, d\tau,$$

---

*Depending on the value of the mutual inductance M, the considered oscillator represents, as it is shown [see (13), (14)], not only an electronic model of the 1st circuit (with the restriction (7)), but, as it can easily be seen, the 2nd circuit in B. V. Bulgakov's problem [2]. Here, $M < L/\mu$ corresponds to a stable object with positive self-recovery, while $M > L/\mu$ corresponds to an unstable (with negative self-recovery) one.

**For a sufficiently small $|h|$: $|f(x + h) - f(x)| <$ const h. These Lipshits conditions can be replaced by Osgoode's conditions, or other less restricting ones, guaranteeing single-valuedness and continuous dependence on the initial conditions.

***$f|[\theta x_1 + (1 - \theta) x_2]| \leq \theta |f(x_1)| + (1 - \theta)|f(x_2)|$ when $0 \leq \theta \leq 1$ for all $|x_1|$ and $|x_2|$.

where

$$\lambda^2 = 4k_\gamma - \rho_2^2 = 3k_\gamma + \rho_2\left(\frac{k_\gamma}{\rho_2} - \rho_2\right) > 0$$

and

$$C_1 = y(0), \quad \lambda C_2 = \rho_2 y(0) + 2z(0).$$

Integrating by parts, we obtain

$$y = \left\{y(0)\cos\frac{\lambda}{2}t + \frac{1}{\lambda}\left[\rho_2 y(0) + 2z(0) - 2k_\gamma x(0)\right]\sin\frac{\lambda}{2}t\right\}\exp\left(-\frac{\rho_2}{2}t\right) + \\ + k_\gamma\int_0^t x(\tau)\left\{\left[\cos\frac{\lambda}{2}(t-\tau) - \frac{\rho_2}{\lambda}\sin\frac{\lambda}{2}(t-\tau)\right]\exp\left[\frac{\rho_2}{2}(\tau - t)\right]\right\}d\tau. \tag{17}$$

Differentiating both parts and noting that $\dot{y} = z$, we obtain

$$z = \left\{\left[z(0) - k_\gamma x(0)\right]\cos\frac{\lambda}{2}t + \frac{1}{\lambda}\left[k_\gamma\rho_2 x(0) - 2k_\gamma y(0) - \rho_2 z(0)\right]\times \\ \times \sin\frac{\lambda}{2}t\right\}\exp\left(-\frac{\rho_2}{2}t\right) + k_\gamma x(t) - k_\gamma\int_0^t x(\tau)\left\{\frac{2k_\gamma - \rho_2^2}{\lambda}\sin\frac{\lambda}{2}(t-\tau) + \\ + \rho_2\cos\frac{\lambda}{2}(t-\tau)\right]\exp\left[\frac{\rho_2}{2}(\tau - t)\right]\right\}d\tau. \tag{18}$$

Let us suppose that the starting point X(0) belongs to a sphere C, with center at the origin and radius r, i.e., $X(0) \in C$, and let us examine, in the time interval $[0, \tau]$ the restricting inequalities resulting from (17) and (18)

$$|y(t)| < mr + k\bar{x}, \quad |z(t)| < nr + l\bar{x}, \tag{19}$$

where

$$m = 1 + 2k_\gamma + \frac{2+\rho_2}{\lambda}, \quad k = 2k_\gamma\left(\frac{1}{\lambda} + \frac{1}{\rho_2}\right),$$

$$n = 1 + k_\gamma + \frac{k_\gamma\rho_2 + 2k_\gamma + \rho_2}{\lambda}, \quad l = k_\gamma\left(3 + \frac{\lambda}{\rho_2} - \frac{\rho_2}{\lambda}\right), \quad \bar{x} = \max|x(t)|\, 0 \leqslant t \leqslant \tau.$$

Let us suppose that the class A function $f(x)$ satisfies the following "dissipativity" conditions. There exists a sufficiently large $r > 0$ such that the roots of the equations

$$m\omega + k\xi = f(\xi), \quad n\omega + l\eta = f(\eta), \tag{20}$$

$\xi, \eta > 0$, corresponding to $\omega = r$, and $\underline{\xi}$, $\underline{\eta} < 0$ when $\omega = -r$, lie in the interval $[-r, r]$ and moreover, the following inequalities are satisfied:*

$$-\frac{r}{k} < \underline{\xi} < \xi < \frac{r}{k}, \quad -\frac{r}{l} < \underline{\eta} < \eta < \frac{r}{l}. \tag{21}$$

---

*Inequalities (21) are satisfied, for example, for the function $f(\xi)$, for which $f'(\xi) \to +\infty$ when $\xi \to +\infty$. In fact, applying l'Hospital's rule, we obtain

$$\lim \xi / r = \lim m / [f'(\xi) - n] = 0 \text{ when } r \to +\infty,$$

Then the following inequalities are fulfilled:

$$\underline{\xi} < x(t) < \bar{\xi} \text{ when} t > 0, \tag{22}$$

only if $\underline{\xi} < x(0) < \bar{\xi}$. If this is not fulfilled then two cases can arise. In the first there is a minimum $\tau > 0$, for which $x(\tau) = \xi$ and $\dot{x}(\tau) \geq 0$. However, in this case, a contradiction results because

$$\dot{x}(\tau) = y(\tau) - f[x(\tau)] < mr + k\xi - f(\xi) = 0.$$

A similar contradiction arises in the second case when there exists a minimum $\tau > 0$ for which $x(\tau) = \underline{\xi}$ and $\dot{x}(\tau) \leq 0$.

From Inequalities (19) and (22) it follows that every trajectory of (16), passing at time $t = 0$, through the point X(0), lying within parallelipiped (15), is Lagrange-stable in the positive sense, since every such positive semi-trajectory ($t > 0$) is within the sphere S with radius

$$R = r\left[(m+1)^2 + (n+1)^2 + \left(\frac{1}{k}\right)^2\right]^{\frac{1}{2}} * .$$

Let us now prove the existence of a quantity $T > 0$ for which the trajectory $X[t, X(0)] \in P$ when $t > T$, if $X(0) \in P$.

There exists a number $\epsilon > 0$ which is such that the inequalities

$$\begin{gathered} -\frac{r-\epsilon}{k} < \underline{\xi} < \xi < \frac{r-\epsilon}{k}, \\ -\frac{r-\epsilon}{l} < \underline{\eta} < \eta < \frac{r-\epsilon}{l}, \end{gathered} \tag{23}$$

are satisfied, which is a consequence of (21).

Let us denote $T = \max(T_1, T_2)$, where

$$T_1 = \frac{2}{\rho_2} \ln \frac{mr}{\epsilon}, \quad T_2 = \frac{2}{\rho_2} \ln \frac{nr}{\epsilon}.$$

From a more accurate evaluation of expressions $\underline{y}$ and $\underline{z}$ than in (19), there result from (17) and (18) the following restricting inequalities:

$$\begin{gathered} -mr\exp\left(-\frac{\rho_2}{2}t\right) + k\underline{\xi} < y(t) < k\xi + mr\exp\left(-\frac{\rho_2}{2}t\right), \\ -nr\exp\left(-\frac{\rho_2}{2}t\right) + l\underline{\eta} < z(t) < l\eta + nr\exp\left(-\frac{\rho_2}{2}t\right). \end{gathered}$$

From Inequalities (22) and (23) it follows that

$$-r \leqslant \xi < x(t) < \xi \leqslant r, \quad |y(t)| < r, \quad |z(t)| < r \text{ when } t > T, \tag{24}$$

if $\quad \underline{\xi} < x(0) < \bar{\xi}, \quad |y(0)| \leqslant r, \quad |z(0)| \leqslant r.$

---

*For Lagrange-stability it is necessary that only Condition (22) be fulfilled, which is guaranteed, for example, only by the first Inequality in (21). All the other formulae containing $\underline{z}$ [(18) and further] are, in this case, superfluous. Indeed, from (16) it follows that $\ddot{y} = -\rho_2\dot{y} - k_\gamma f(x)$. From this $|\ddot{y}| < \rho_2 \; |\dot{y}| + k_\gamma |f(x) < \rho_2 \dot{y}^2 + (\rho_2 + k_\gamma \bar{f})$, where $= \bar{f}$ $\max f(x)$. According to Tonelli's lemma [19] $z = \dot{y}$ is bounded together with $\underline{y}$.
$\underline{\xi} \leq x \leq \xi$

3. Liapunov-Instability of the Zero Solution

Certain local properties of the zero solution of System (12) will be needed later on, in particular, its repulsiveness (repulsion of the trajectory in certain special neighborhoods of the zero solution).

For the sake of simplicity, let us restrict our examination to those class A functions which, in the neighborhood of zero, are represented in the form

$$f(x) \equiv Dx + \varphi(x), \tag{25}$$

where the constant D satisfies the repulsiveness condition

$$0 < D < \beta_2 - \rho_2, \tag{26}$$

while the function $\varphi(x)$ satisfies Bellman's condition [20]*

$$\varphi(x) = o(x). \tag{27}$$

In this case, the linear approximation of (12) is the system

$$\begin{aligned} \dot{x} &= y - Dx, \\ \dot{y} &= z, \\ \dot{z} &= -\rho_2 z - k_\gamma Dx, \end{aligned} \tag{28}$$

the characteristic equation of which

$$\lambda^3 + (D + \rho_2)\lambda^2 + \rho_2 D\lambda + k_\gamma D = 0 \tag{29}$$

has an unstable zero solution, because its Hurwitz determinant is

$$H = -\rho_2 D(\beta_2 - \rho_2 - D) < 0.$$

On the other hand, when the repulsivity Condition (26) is satisfied H > 0, and the zero solution of (27) is stable.

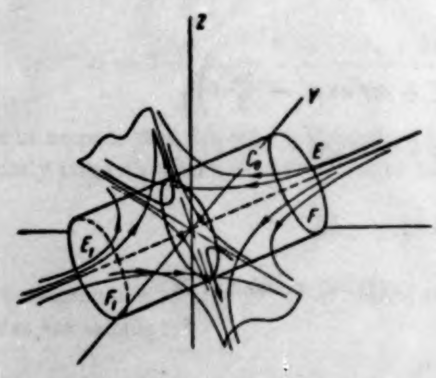

Fig. 4.

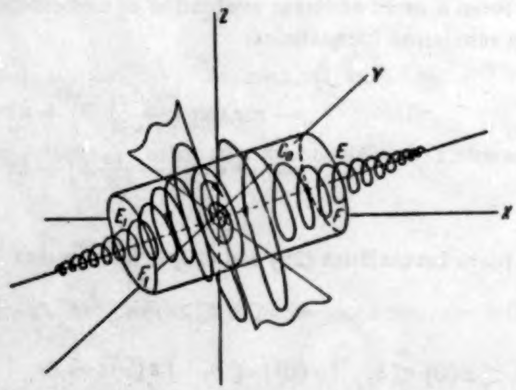

Fig. 5.

It is known (direct and reverse theorems in Liapunov's second method) that, under Bellman's Condition (27), the local stability and instability properties of the zero solution of the linear approximation (28) are carried over to the zero solution of the nonlinear System (12).

---

* o(x) denotes a quantity of an order of smallness greater than the order of smallness of x.

Under Conditions (26), only two cases are possible for the roots of the characteristic equation of (28), to the first of which correspond the three real roots λ, μ, ν, while to the second, one real and a complex conjugate pair λ, $\mu + i\nu$, $\mu - i\nu$. The values λ, μ, ν are, in both cases, positive.

In the first case, the origin is an unstable, singular (particular) point of the "node-saddle" type (Fig. 4). System (12) is reduced to the form

$$\dot{x} = \mu x + \epsilon_1, \quad \dot{y} = \nu y + \epsilon_2, \quad \dot{z} = -\lambda z + \epsilon_3,$$

where ϵ_1, ϵ_2 and ϵ_3 are an order of magnitude smaller than $(x^2 + y^2 + z^2)^{1/2}$.

From here, in a sufficiently small vicinity of the origin there is an open cylinder C_0, center at the origin, radius $\rho_0 = (x^2 + y^2)^{1/2}$ and height $\overline{E'E}$ (Fig. 4), which is such that all the trajectories emerge from its side surfaces, since $\frac{d}{dt}(x^2 + y^2) = 2(\mu x^2 + \nu y^2) + 2(x\epsilon_1 + y\epsilon_2) > 0$, and enter thru its bases F and F_1 (faces), since there $\frac{d}{dt}z^2 = -2\lambda z^2 + 2z\epsilon_3 < 0$. The cylinder's vertical axis $\overline{E'E}$ lies along the straight line

$$y = (D - \lambda)x, \quad z = -\lambda y, \tag{30}$$

where $D - \lambda = -\rho_2 - \mu - \nu < 0$, passing through the first quadrant ($x > 0$, $y < 0$, $z > 0$) when $x > 0$, and through the second ($x < 0$, $y > 0$, $z < 0$) when $x < 0$. From this it follows that the diameter of the region containing C_0 can be made so small that the base F will lie in the first, and F_1 in the second quadrants, while the projections of these bases $F_{z=0}$ and $F_{1z=0}$ onto the planes $z = 0$ will lie, correspondingly, in the fourth and second quadrants of the plane $z = 0$.

In the second case of the location of the roots ($-\lambda$, $\mu + i\nu$, $\mu - i\nu$) the picture is analogous to the first case (Fig. 5). The origin is an unstable, singular point of the "focus-saddle" type. In a sufficiently small vicinity of the origin there is a cylinder C_0 with properties similar to those of the first, and the bases of which F and F_1 are also situated, respectively, in the first and second quadrants, while the projections $F_{z=0}$ and $F_{1z=0}$ are in the fourth and second quadrants of the plane $z = 0$, since, in this case too, $D - \lambda = -\rho_2 - 2\mu < 0$.

Cylinder C_0 will play an important part later on.

4. Separately Generated Surfaces of the Phase Portrait

In phase space, let us examine two cylindrical surfaces S and S_1 (Fig. 6) with generators parallel, respectively, to the axes $\overline{OZ}$ and $\overline{OY}$, and directrices the flat curves $\Gamma[y = f(x)$ (in the plane $x = 0$)] and $\Gamma_1[z = -k_\gamma/\rho_2\ f(x)$ (in the plane $y = 0$)].

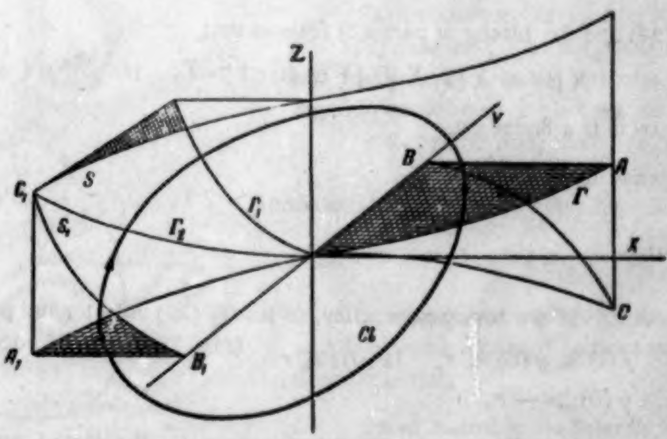

Fig. 6.

The line of intersection of S_1 with the plane $z = 0$ is a straight line – the axis $\overline{OY}$, while that of S_1 with S is a space curve $\Gamma_2[y = f(x),\ z = -k_\gamma/\rho_2\ f(x)]$.

Let us examine the trihedral, open, infinite region C_1, bounded by the surfaces S, S_1, the plane $z = 0$ and the edges $\Gamma^{+}[y = f(x)$ when $x > 0]$, $\Gamma_2^{(-)}\left[y = f(x),\ z = -\frac{k_Y}{\rho_2} f(x)\right.$ when $\left.x < 0\right]$, and the negative half of the axis $\overline{OY}$ ($y < 0$). Figure 6 shows part of the region C_1 $\{\breve{OA}, \breve{OC_1}, \overline{OB_1}\}$.

In the region C_1 the derivatives $\dot{x}$, $\dot{y}$, $\dot{z}$ do not change sign. If the trajectory $X(t) = X[t, X(0)]$, at time $t = 0$, shares a common point X(0) with a boundary of C_1 (the origin is excluded from the investigation), i.e., $X(0) \in \bar{C_1}/C_1$*, $X(0) \neq \{0, 0, 0\}$, then X(t) leaves C_1 strictly when $t > 0$.**

Indeed, if $X(0) = \{x(0), y(0), z(0)\}$ lies on some edge of region C_1, for example on boundary S_1, then the projection of the velocity vector

$$\dot{X}(0) = \{\dot{x}(0), \dot{y}(0), \dot{z}(0)\} \text{ on grad } S_1 = \{-k_Y f'[x(0)],\ 0,\ -\rho_2\}\ ^{*}$$

will be equal to

$$X(0)\,\mathrm{grad}\,S_1 = -k_Y f'[x(0)]\{y(0) - f[x(0)]\} > 0;$$

if X(0) lies on an edge of region C_1, for example on $\Gamma_2^{(-)}$, then $X(0) \times \mathrm{grad}\, S_1 = 0$, but $\dot{X}(0)\ \mathrm{grad}\ S = z > 0$ ($\mathrm{grad}\ S = \{-f'[x(0)], 1, 0\}$).

Consequently, trajectory X(t), in both cases, leaves C_1 when $t > 0$. The proof is analogous in other cases. From this it follows that closed trajectories cannot be contained in region C_1.

Analogous examinations of an, in a sense, symmetrical (relative to the plane $\{\overline{OY}, \overline{OZ}\}$) with respect to C_1**** trihedral, open, infinite region C_{11}, bounded by the surfaces S, S_1, the plane $Z = 0$ and edges $\Gamma^{(-)}$, $\Gamma_2^{(+)}$ and the positive half of the axis $\overline{OY}$, permit one to conclude that for C_{11} too, trajectories having, at $t = 0$, common points with its boundary, leave C_{11} when $t = 0$. Moreover, region C_{11} also does not contain any closed trajectories.

For what follows it is important to note that not a single trajectory $X(t) = X[t, X(0)]$, $X(0) \bar{\in} C_1, C_{11}$ ($X(0) \neq 0$)***** can adjoin the origin, i.e., $|X(t)| = (x^2 + y^2 + z^2)^{1/2} \to 0$ when $t \to +\infty$, because the straight line (30) is contained, together with the bases (end faces) F and F_1 of cylinder C_0, in the regions C_1 and C_{11}.

5. Proof of Existence of Cycle

Let us examine, in the three-dimensional phase space E, the sum of the open regions C_1, C_{11}, C_0 (open cylinder, part 3): $\Omega_1 = C_1 \cup C_{11} \cup C_0$, an additional, to this sum, closed region Ω, and the intersection of the latter with a closed parallelipiped $\bar{P}$ (P + boundary, part 2) – a closed toroidal-shaped region $\bar{\omega} = \bar{\Omega} \cap \bar{P}$ (Fig. 7).

From Inequality (24) and the results of part 4 it follows that

$$X(t) = X[t, X(0)] \in \omega \text{ when } t > T, \quad \text{if} \quad X(0) \in \omega, \tag{31}$$

where the open region $\omega = \bar{\omega}$ is a boundary.

Moreover, let us prove that

$$X(t) \in \omega \text{ when } t > 2T, \tag{32}$$

if $X(0) \in \bar{\omega}$.

For this it is sufficient, without losing generality, to justify (32) only for the points X(0) lying on the edge MNVW, where $x(0) = \xi$, $f(\xi) \leqslant y(0) \leqslant r$, $|z(0)| \leqslant r$ (Fig. 7), or on the edge $M_1N_1V_1W_1$, where

$x(0) = \underline{\xi}$, $f(\underline{\xi}) \geqslant y(0) \geqslant -r$,

*The slanted line / indicates the difference between two regions, $\bar{C_1}$ the closed, and C_1 the open regions.

**I.e., not one trajectory can touch the boundaries of C_1.

*** At the points of discontinuity $f'(x) = f'(x + 0)$ or $f'(x - 0)$ (for simplicity we will assume $f'(x)$ to be piecewise continuous with 1st kind discontinuities).

****Since in the functional class A $xf(x) > 0$ when $x \neq 0$.

*****X(0) does not belong to the regions C_1 and C_{11}.

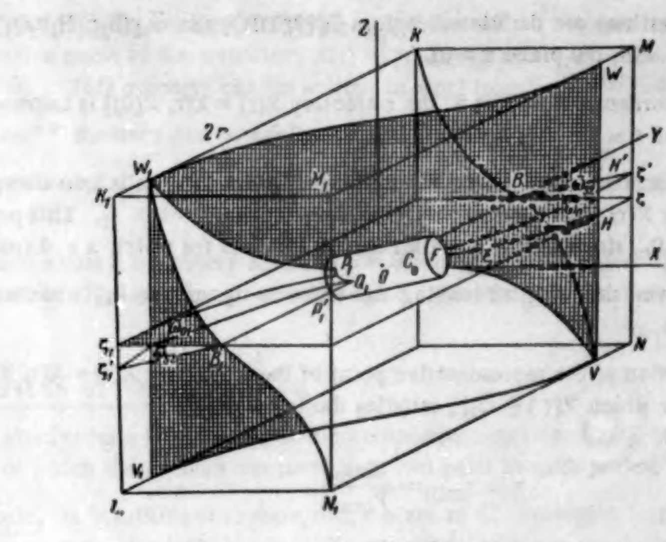

Fig. 7.

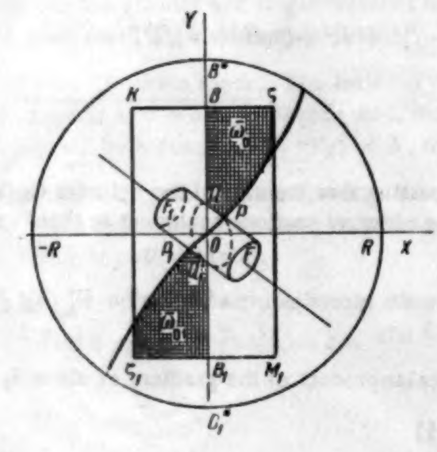

Fig. 8.

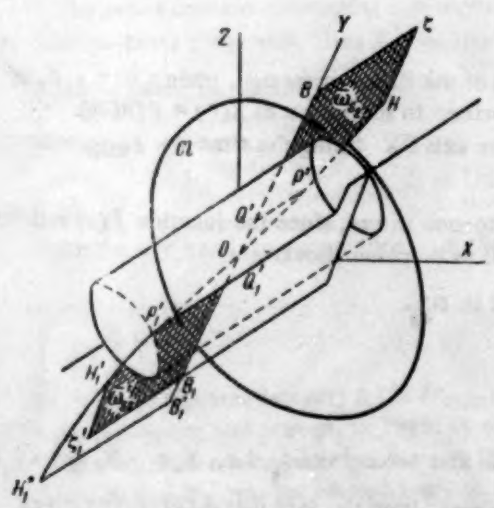

Fig. 9.

Trajectory $X[t, X(0)]$, emerging from the point $X(0)$ of the boundary $MNVW$ ($M_1N_1V_1W_1$), leaves $\overline{\omega}$ for sufficiently small value of $t > 0$ and, consequently, cannot return to this boundary for sufficiently large values of $\underline{t}$.

For a sequence of interval with respect to $\overline{\omega}$, points $\{X_n(0)\}_{n=1,2,\cdots}$, (i.e., $X_n(0) \in \omega$), for which $X_n(0) \to \lambda(0)$ when $n \to \infty$, it follows from the continuity of $f(x)$ that for each fixed number $\theta \geq 0$

$$X[T+\theta,\ X_n(0)] \to X[T+\theta,\ X(0)] = \overline{X}(0).$$

From (31) it follows that, for each $\underline{n}$, the point $X[T+\theta + X_n(0)] \in \omega$. Consequently, $X(0) \in \overline{\omega}$, and since $X(0)$ cannot belong to the boundaries $MNVW$ and $M_1N_1V_1W_1$, then, taking (24) into account, Relationship (31) is applicable to this point. From this follows (32), which was what was required to prove.

Each trajectory $X(t) = X[t, X(0)]$, for which $X(0) \in \overline{\omega}$ and $X(0)$ does not lie in the plane $z = 0$, intersects the latter when $t > 0$. In the opposite case a contradiction arises because when $\dot{y} = z$ does not change sign, $\underline{y}$ is bounded ($|y| < R$, part 2) and, therefore, there exists $\lim\limits_{t\to\infty} y(t) = y_\infty$ which determines the other boundaries, following from (12), x_∞ [moreover, $y_\infty = f(x_\infty)$], $z_\infty = -\dfrac{k\gamma}{\rho_2}\, y_\infty$.

In accordance with (32), the singular point $\{x_\infty, y_\infty, z_\infty\} \in \overline{\omega}$, which cannot be because $\overline{\omega}$ does not contain singular points.

Let us examine the intersection of the sphere S (part 2) with the plane $z = 0$ (Fig. 8).

In the circle S_0, with radium R, there is a rectangle $K\zeta M_1 \zeta_1$ (the intersection of the plane $z = 0$ with the parallelipiped $\overline{P}$) and the closed regions $\overline{\omega}_0(B^* H^* PQ)$ and

$\Omega_{01}(B_1^* H_1^* P_1 Q_1)$. Inside these last two are the closed regions $\bar{\omega}_0 (B\zeta HPQ)$ and $\bar{\omega}_{01}(B_1\zeta_1 H_1 P_1 Q_1)$ (intersection of the toroidal-shaped, closed region $\bar{\omega}$ with the plane z = 0).

If $X(0) \in \bar{\omega}_{01}$, then, in accordance with part 2, the trajectory $X(t) = X[t, X(0)]$ is Lagrange-stable in the positive sense, i.e., $X(t) \in S$ when $t > 0$.

Moreover, for sufficiently small $t = t_1 > 0$, the representative point $X(t_1)$ falls into that part of $\bar{\omega}$, where $z > 0$ and, consequently, the trajectory X(t) must intersect the plane z = 0 when $t = t_2 > t_1$. This point of intersection $X_2 = X(t_2)$ must lie in the region Ω_0, since into it enter all the trajectories for which $z > 0$ (part 4).

Analogously, it can be proved that X(t), in leaving $\bar{\Omega}_0$ when $t > t_2$, crosses Ω_{01} when $t = \tau > t_2$, i.e., $X(\tau) \in \bar{\Omega}_{01}$.

The time τ for one revolution of the representative point of the trajectory $X(t) = X[t, X(0)]$, $[X(0) \in \bar{\omega}_{01}]$ about the coordinate axis $\overline{OX}$, for which $X(\tau) \in \bar{\Omega}_{01}$, satisfies the inequality*

$$\tau > \tau_{\min} = \frac{2\pi\rho_0}{V_{\max}}, \tag{33}$$

where ρ_0 is the radius of the base of the cylinder C_0 (part 3) while

$$V_{\max} = \max_{X \in \bar{S}} |\dot{X}(t)| = \max(\dot{x}^2 + \dot{y}^2 + \dot{z}^2)^{\frac{1}{2}} < [(R + \bar{f})^2 + R^2 + (\rho_0 R + k_\gamma \bar{f})^2]^{\frac{1}{2}},$$

$$\bar{f} = \max\{|f(\xi)|, f(\xi)\}.$$

If one examines the section of sphere S made by the plane S_2, passing thru the axis of the cylinder C_0 [the axis lies on the straight line (30] and the coordinate axis $\overline{OY}$, then one observes sections analogous to those examined earlier (Fig. 9).

The region $\bar{\Omega}'_{S_2}$ $(B^*_1 H_1^{**} P_1' Q_1')$ is analogous to region $\bar{\Omega}_{01}$, while the closed convex** region $\bar{\omega}'_{S_2}(B_1\zeta_1' H_1' P_1' Q_1')$ is analogous to ω_{01}.

There are no trajectories touching the region $\bar{\Omega}'_{S_2}$ because the scalar product of the gradient of plane S_2

$$\operatorname{grad} S_2 = \{\lambda(D-\lambda), 0, 1\}$$

with the velocity vector of the representative point $V = \{\dot{x}, \dot{y}, \dot{z}\}$

$$V \operatorname{grad} S_2 = \lambda(D-\lambda)[y - f(x)] - [\rho_2 z + k\, f(x)] > 0$$

for all points $X(0) \in \bar{\Omega}'_{S_2}$.

For each point $X(0) \in \bar{\omega}'_{S_2}$ there is a corresponding point $X(\tau)$ of the first intersection, when $t = \tau > 0$, of the trajectory $X(t) = X[t, X(0)]$ with the region $\bar{\Omega}'_{S_2}$, which can be written in short form as $X(\tau) = F[X(0)]$. In this expression, point X (t) corresponds to one revolution about the axis $\overline{OX}$ during the time $\tau > \tau_{\min}$, where $\tau_{\min}$ is determined by (33).

The established correspondence F is continuous and of a one-to-one nature, since the function $f(x)$ satisfies the Lipshits conditions. Such correspondences are called topological or homomorphous.

Region $\bar{\omega}'_{S_2}$ is made to correspond to region $F(\bar{\omega}'_{S_2})$, contained in $\bar{\Omega}'_{S_2}$.

---

*For each t in the interval $0 < t < \tau$, X(t) does not intersect $\bar{\Omega}_{01}$.

**If $X_1, X_2 \in \bar{\omega}'_{S_2}$, then each point of the segment connecting them also belong to $\bar{\omega}'_{S_2}$, i.e., $X_1\xi_1 + X_2\xi_2 \in \bar{\omega}'_{S_2}$, where the positive quantities ξ_1 and ξ_2 satisfy $\xi_1 + \xi_2 = 1$. This follows from the fact that $f(x)$ is nonconvex while the section ($P_1'Q_1'^*$, Fig. 8) of the plane S_2 with the curved side of cylinder C_0 is a straight segment.

In accordance with (32), for each point $X(0) \in \bar{\omega}'_{S_2}$, there will be required not more than $N = 1 + E\,(2T/\tau_{min})$* turns of the representative point of the trajectory $X(t) = X[t, X(0)]$ about the OX axis, for the latter to arrive back, moreover into $\bar{\omega}'_{S_2}$. This property can be written in short form as: $F^{(N)}(\bar{\omega}'_{S_2}) \quad \omega'_{S_2}$.

The Bol'-Brouwer** theorem can be applied to the convex figure $\bar{\omega}'_{S_2}$.

In the continuous mapping of a convex figure into itself, there exists a stationary point [22].*** The latter determines the cycle Cl.

In this way, there exists a trajectory $X(t)$ and the quantity $T_C > 0$, called the period, which is such that $X(t + T_C) = X(t)$ for all t. The question of whether there exist other cycles, besides Cl, for System (12) remains open.

6. Some Properties of the Cycle

The cycle is called simple if the zeros of the component functions $\{x(t), y(t), z(t)\}$ are devisible by the zeros of their derivatives, of which there exists not more than two pairs in each period T_C.

Cycle Cl -simple. It is sufficient to show that the arc of Cl emerging from the point $X(0) \in \omega'_{S_2}$, returns for the first time into $\bar{\omega}'_{S_2}$ to the point $X(T_C) = X(0)$ having made only one revolution about the OX axis, and not $N > 1$, as it has already been established. The latter is guaranteed by the Levinson-Massera theorem [23, 18]: if in the topological (continuous and single-valued) mapping F of the flat figure $\bar{\omega}'_{S_2}$ into itself $F^{(n)}(\bar{\omega}'_{S_2}) \quad \omega'_{S_2}$ for every $n > N$, then there is a stationary point.

Cycle Cl-stable mode. The trajectory $X[t, X(0)]$ ($= X(0)$ when $t = 0$) is called a stable mode if, for each $\epsilon > 0$, there is a $\delta = \delta(\epsilon) > 0$ such that, for any pair of points X_1 and X_2 on the trajectory which, at time $t = t_0$, are separated by a distance $|X_1 - X_2| < \delta$, the distance $|X(t, X_1) - X(t, X_2)| < \epsilon$ for any t ($t \le t_0$, $t \ge t_0$).

In the theory of dynamic systems, this "internal stability" property was first formulated by Franklin, who called it the S-property. The zero solution of System (12) $X[t, o)$ possesses this S-property and is therefore a stable mode which is called trivial.

The cycle Cl $X(t + T_C) \equiv X(t)$ is a nontrivial stable mode.**** In the opposite case there exists series of points on Cl, $\{X_{1n}\}_{n=1,2,\dots}$ $\{X_{2n}\}_{n=1,2,\dots}$ and $\{t_n\}_{n=1,2,\dots}$ $0 \le t_n \le T_C$ and the number $\epsilon > 0$ such that

$$|X(t_n, X_{1n}) - X(t_{1n}, X_{2n})| \geqslant \varepsilon$$

and

$$|X_{1n} - X_{2n}| \to 0 \text{ when } n \to \infty. \tag{34}$$

The series contain converging sub-sequences $\{X'_{1n}\}_{n=1,2\dots}$, $\{X'_{2n}\}_{n=1,2\dots}$ and $\{t'_n\}_{n=1,2,\dots}$, possessing the same property (34), with $\lim\limits_{n\to\infty} X'_{1n} = \lim\limits_{n\to\infty} X'_{2n} = X_\infty$, $\lim\limits_{n\to\infty} t'_n = t_\infty$, $0 \leqslant t_\infty \leqslant T_c$.

Let us examine the limit

$$L = \lim_{n\to\infty} |X(t'_n, X'_{1n}) - X(t'_n, X'_{2n})|.$$

On the one hand, in accordance with (34) $L > 0$, on the other hand,

---

*The function (greater integer) $E\,(2T/\tau_{min})$ is equal to the smallest whole number enclosed in $2T/\tau_{min}$.

**Such a theorem was proven, in 1905, by the Latvian mathematician P. R. Bol' and, independently of him, in 1911 by the Dutch mathematician Brauwer [21].

***The region $\bar{\omega}'_{S_2}$ was introduced into the investigation for the sake of clarity of discussion. It can be shown that the region $\bar{\omega}_{01}$ is homomorphous to a circle and, because $F^{(N)}(\bar{\omega}_{01}) \quad \bar{\omega}_{01}$, Bol'Brauwer's theorem is applicable to it too.

****Periodic, nontrivial, stable modes also exist.

$$L = \lim_{n\to\infty}[X(t_\infty, X'_{1n}) - X(t_\infty, X'_{2n})] + \lim_{n\to\infty}[X(t'_n, X'_{1n}) - X(t_\infty, X'_{1n})] + $$
$$+ \lim_{n\to\infty}[X(t_\infty, X'_{2n}) - X(t'_n, X'_{2n})] = 0,$$

since the trajectory Cl $X[t, X(0)]$ is continuous with respect to $X(0)$ for a fixed t, and is continuous with respect to t for a fixed $X(0)$. The latter follows from the continuity of $f(x)$.

The obtained contradiction proves that (34) is impossible, and, therefore, Cl is a stable mode. The question of the "external" stability of the trajectory of the Cl cycle with respect to other transient modes remains open, as for example, the problem of the Liapunov stability of Cl, in particular its assumptotic stabilities and its regions of attraction [24].

The method set forth permits one to investigate the behavior of a three-dimensional, automatic control system, with one nonlinear control element, in phase space. However, the results can be extended to include three-dimensional and multi-dimensional systems with several different types of nonlinearity.

The relationships presented permit one, in practical cases, to evaluate the adjustment limits of the transfer ratios of automatic machines, corresponding to different operating modes of the system, as well as the limiting values of the amplitude of possible oscillations.

Particularly simple results are obtained for the piecewise-linear characteristics of a control element.

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CONCERNING THE NOISE STABILITY OF PULSE-FREQUENCY TELEMETRY

N. V. Pozin

Abstract

The real noise stability of pulse-frequency telemetering, which takes into account the basic methods of realizing means of transmitting information, is analyzed. The mean error and the mean-square error are taken as criteria of noise stability. The method of signal "discretization" is used in the analysis. Design formulae for the evaluation of noise stability are deduced.

The noise stability of telemetering can be characterized quite completely by means of the mean and mean-square errors. In pulse-frequency systems utilizing pulse frequency meters,* these errors can be conveniently determined by means of a mean number of false pulses, produced by interference in unit time.** This number characterizes the mean deviation of readings on the receiving apparatus from the actual value in the absence of noise, i.e., the mean error of telemetering. The mean-square error is determined from the mean square of deviations from the mean.

Let us consider the problem of finding these errors by examining the low-frequency channel of a receiving device under the following conditions:

1) the duration of the received pulses is equal to that of the spaces (i.e., the mark-space ratio of the pulses is two), with the pulse repetition frequency being between f_1 and f_2;

2) fluctuating (smooth) noise acts in the channel;

3) the low-frequency channel of the receiving unit contains a low-pass filter with an upper cut-off frequency f_{co} ($f_{co} \geq f_2$), a bilateral clipper which cuts out from the incoming pulses a narrow band at a U_0 (Fig. 1), a pulse frequency meter which responds to arbitrarily shaped pulses with an upper repetition frequency $f_u \geq f_{co}$.

1. Analysis of Noise Stability by the Method of Signal "Discretization"

To analyze the interaction of a signal with noise let us divide the pulse repetition period $1/f$ (Fig. 1) into intervals of time $\Delta t = 1/2f_{co}$ and let us call Δt the duration of a unit (elementary) time segment. Let us designate the number of unit time segments in a pulse or a space by

*For example, capacitor frequency meters, etc.

**We are concerned here with the determination of errors, dependent on relatively strong noise, in the presence of which the initial distortion of the signal occurs at the expense of the division of pulses and spaces. Correlation methods of reception are not examined. The effect of weak, fluctuating noise, causing fluctuation of the leading edges, and introducing some very small mean-square error component, is not taken into account.

$$n = \frac{f_{co}}{f}, \tag{1}$$

where f is the pulse repetition frequency.

Let us apply the term unit (elementary) distortion to the false space of duration Δt, formed at the output of the limiter, under the action of noise, in any part of the transmitted pulse, or to the false pulse, of the same duration, in any part of a space.

For simplicity, let us assume that the probability of unit distortion in the space P_s and the pulse P_p are equal, i.e.,

$$P_s = P_p = P. \tag{2}$$

Let us suppose that the probability P of unit distortion is known.* Because the above defined quantity Δt represents the correlation interval of smooth noise at the output of the filter, the formation of unit distortions can be considered to be independent [1]. The probability P(m) of m out of n unit elements of time being distorted can be found by means of, for example, the binomial law of probability distribution

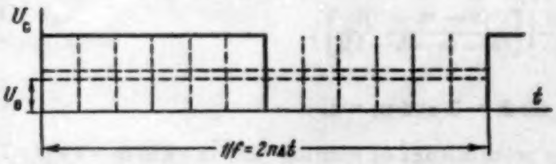

Fig. 1. Break-down of pulse repetition period into 2n segments of time, each of duration $\Delta t = 1/2f_{co}$.

$$P_m = C_n^m P^m (1 - P)^{n-m}, \tag{3}$$

where

$$C_n^m = \frac{n!}{m!\,(n-m)!}$$

is the number of combinations of n taken m at a time.

Let us examine a half-period. If there are m unit distortions, then there can be different combinations of these, resulting, for example, only in a change in the shape of the transmitted pulse, if the end regions of the pulse are distorted (spaces), or in the break-down of the transmitted pulse into 2, 3, . . ., (m + 1) parts, each of which has a duration of not less than Δt. This break-down of the transmitted pulse is received by the receiver as the formation of 1, 2, . . ., m false pulses.

To find separately the probabilities of the formation of various numbers of false pulses for m unit distortions, let us represent the probability P_m as a sum of probabilities $P_m(k)$, where k is the number of false pulses:**

$$P_m = \sum_{k=0}^{m} P_m(k). \tag{4}$$

Determining $P_m(k)$ as

$$P_m(k) = A_m(k)\, P^m (1 - P)^{n-m}, \tag{5}$$

we represent the number of combinations C_n^m as a sum of coefficients $A_m(k)$:

$$C_n^m = \sum_{k=0}^{m} A_m(k). \tag{6}$$

*For the determination of P_p and P_s (or P if $P_p = P_s$) for some cases see the Appendix.

**The method for calculating the mean number $\bar{k}$ of false pulses in a half-period, presented below, does not take into account the possible distortions in the edges of the pulse and space, due to which, distortions in the shape of the pulse (space) can become false pulses. However, taking into account the relatively small significance of the combinations not taken into account, this method is preferable from a point of view of simplicity and objectivity.

From an examination of the combinations of C_n^m it is simple to find that the coefficient $A_m(0)$, which shows how many combinations lead to a change in the shape of the transmitted signal, and the coefficients $A_m(1)$ and $A_m(2)$, which indicate how many combinations lead to the formation of one and two false pulses, are determined from the formulae

$$A_m(0) = m + 1,$$

$$A_m(1) = \frac{1}{2}[(m+1)m][n-m-1], \tag{7}$$

$$A_m(2) = \frac{1}{12}[(m+1)m(m-1)][(n-m-1)(n-m-2)].$$

It can be shown that, in the general case, the number of combinations which form k false pulses for m unit distortions out of a possible n, is determined by the formula

$$A_m(k) = \frac{k+1}{[(k+1)!]^2}\left[\frac{(m+1)!}{(m-k)!}\right]\left[\frac{(n-m-1)!}{(n-m-1-k)!}\right], \tag{7a}$$

which is valid for all positive $k \leq m \leq n-(k+1)$, as well as for $k = -1$ and $m = n$.

The total probability of forming k false pulses, with the probabilities of forming from $m = k$ to $m = n = (k+1)$ unit distortions taken into account, is equal to

$$P(k) = \sum_{m=k}^{m=n-(k+1)} P_m(k). \tag{8}$$

The mean number of false pulses, produced by noise, in a half-period, is equal to

$$k = P(1) + 2P(2) + \ldots + k_m P(k_m) - P(-1) = \sum_{k=-1}^{k_m} kP(k), \tag{9}$$

where k_m is the maximum possible number of false pulses:

$$k_m = \begin{cases} \frac{n-2}{2} & \text{for even} \quad n > 2, \\ \frac{n-1}{2} & \text{for odd} \quad n > 1, \\ -1 & \text{for} \quad n = 1 \text{ and } n = 2. \end{cases}$$

Making use of (8) we obtain

$$\bar{k} = \sum_{k=-1}^{k_m} k \sum_{m=k}^{m=n-(k+1)} P_m(k). \tag{10}$$

Under Condition (2), for each pulse repetition period there is a mean number of false pulses equal to $2\bar{k}$.

It is obvious that $2\bar{k}$ represents the mean, relative error in the readings on the pulse-frequency receiver.

The absolute, mean, telemetering error, due to noise, at a frequency f is equal to

---

* Let us note that the probability of losing the transmitted pulse or of the appearance of a number of false pulses, equal to $k = -1$ is the probability of distorting all $m = n$ unit pulses, i.e., $p(-1) = p^n$.

$$\Delta_{av} = 2f\bar{k}. \tag{11}$$

The value of Δ_{av} effectively depends on frequency, since f appears as a multiplier and $\bar{k}$ is also a function of frequency. In this way, at different points in the frequency range used for telemetering there will be a different mean error due to noise. In telemetry, one is usually interested in some single value of error, reduced to the utilized range $\Delta f = f_2 - f_1$.

Analogously to the accepted method of determination used in telemetry, the reduced mean error can be characterized thru the ratio of the maximum value of the absolute mean error $\Delta_{av\,(max)}$ over the range Δf, to this range, i.e.,

$$\delta_{av} = \frac{\Delta_{av\,(max)}}{\Delta f} \tag{12}$$

The reduced mean error can be defined in yet another way. Let us call the reduced mean error δ_{av} the ratio of the value of the absolute mean error Δ_{av}, averaged over the range Δf, to this range, i.e.,

$$\delta_{av} = \frac{\bar{\Delta}_{av}}{\Delta f} \tag{13}$$

Turning to the determination of the mean-square, telemetering error, let us examine the segment of time T, containing N periods of $1/f$:

$$N = Tf$$

The time T is the persistence of the transmission.*

The absolute mean-square error $\Delta_{me.\,sq.}$ is found from the square root of the dispersion of the distribution of false pulses over time T. The dispersion of this distribution is expressed thru the dispersion of the distribution over the time $t = 1/2f$, i.e., thru the mean value of the square of the deviation of the number of false pulses $\underline{k}$ from the mean number $\bar{k}$. In the case of equal probability of the appearance of false pulses in any period, we have

$$\Delta^2_{me\,sq} = 2N\,\overline{(k - \bar{k})^2} \tag{15}$$

The random quantity $\underline{k}$ can take on values $-1, 0, 1, 2, \ldots, k_m$ corresponding to probabilities $P(-1), P(0), P(1), P(2), \ldots, P(k_m)$.

Let us expand (15) in the form:

$$\Delta^2_{me\,sq} = 2Tf \sum_{k=-1}^{k_n} (k - \bar{k})^2 P(k). \tag{16}$$

Averaging (16) over the range Δf, i.e., obtaining the value of $\bar{\Delta}^2_{me.\,sq.}$ and relating it to the square of the maximum change in the parameter over time T, i.e., to $T^2\ \Delta f^2$, we obtain the square of the reduced mean-square error

$$\delta^2_{me\,sq} = \frac{\bar{\Delta}^2_{me\,sq}}{T^2 \Delta f^2} \tag{17}$$

The above formulae for $P(k)$ and $\bar{k}$, obtained by the method of break-down of a signal into discrete amounts, are valid for integral values of $\underline{n}$. If, when $n > 2\ (f < f_{co}/2)$, the points at which Δ_{av} and $\Delta_{me.\,sq.}$ are calculated are rather closely spaced, then the region $1 < n < 2\ (f_{co} > f > f_{co}/2)$, containing, in many cases, the whole

*It is convenient to consider the time T to be the time for the setting-up of indications in the receiver, if the persistent sections in the transmitting system can be considered to be aperiodic.

operating frequency range, has only two boundary points of calculation. Clearly, the method of signal "discretization" does not yield a rigorous solution within this region.* However, one can make use of a graphical approximation and an experimental verification. Both these methods show that the relationships $\Delta_{av} = \varphi(f)$ and $\Delta^2_{me.sq.} = \psi(f)$, in the region $f_{co}/2$ to f are very nearly linear. Consequently, for the calculation of $\bar{\Delta}_{av}$ and $\bar{\Delta}^2_{me.sq.}$ from Equations (11) and (16) for a series of values of f, with the subsequent determination of the arithmetic mean, as well as for the determination of the simplified equations derived below, one makes use of a linear approximation of the values of Δ_{av} and $\Delta^2_{me.sq.}$ between the calculation points $f_{co}/2$ and f_{co}.

2. Simplified Equations

In practice, great value is attached to a simple expression for the mean number of false pulses in a half-period, representing the approximate equality

$$\bar{k} = (n-2)\,P, \tag{18}$$

obtained from the conditions $\bar{k} \approx A_1(1)P$ and spread over all possible values of $\underline{n}$, i.e., $\infty > n > 1$. Substituting (18) into (11) and making use of (1), we obtain an important, in practice, expression for the mean absolute error due to noise:

$$\Delta_{av} = 2(f_{co} - 2f)\,P. \tag{19}$$

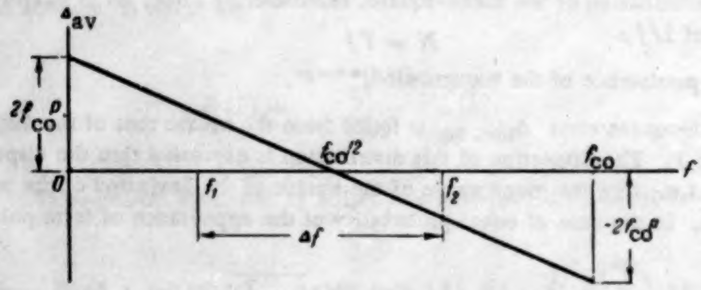

Fig. 2. Distribution of the absolute mean error over the range of frequencies 0 to f_{co}.

Equation (19) gives the value of the absolute error at any point on frequency range 0 to f_{co}. The distribution of the error over this range of frequencies is shown in Fig. 2. The physical explanation for the transition of Δ_{av} into the negative region when $f > f_{co}/2$, lies in the fact that sufficiently short pulses can disappear completely in the presence of noise, but cannot be broken-down into parts.

We integrate the modulus of (19) to obtain the mean over the range, assuming a uniform distribution of the telemetered parameter:

$$\bar{\Delta}_{av} = \frac{1}{\Delta f}\int_{f_1}^{f_2} |\Delta_{av}|\, df. \tag{20}$$

After integration and division by Δf we obtain an expression for the mean, reduced error (13) in the form

$$\delta_{av} = \frac{2P}{\Delta f^2}\left|\left(\frac{f_{co}}{2} - f_1\right)^2 \pm \left(f_2 - \frac{f_{co}}{2}\right)^2\right|. \tag{21}$$

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*Such a solution can, in principle, be obtained by means of the distribution of the duration of the distortions, and the coefficient of correlation of the noise in a video-channel.

In Expressions (21) and (26) the plus is used if the range Δf includes the point $f_{co}/2$, i.e., $f_1 < f_{co}/2 < f_2$, and the minus is used in the opposite case.

If the utilized range Δf is symmetrical about $f_{co}/2$, i.e., if $\frac{f_{co}}{2} - f_1 = f_2 - \frac{f_{co}}{2} = \frac{\Delta f}{2}$, then from (21) we obtain

$$\delta_{av} = P. \tag{22}$$

Let us turn to the expression for the mean-square error. Assuming that the mean deviation $\bar{k}$ is small, taking into account the probability of the formation, in a half-period, of only one false pulse (k = 1) when one unit pulse is distorted (m = 1), and setting $P(k) = A_1(1)P$, we obtain an expression for the sum, which appears as a factor in (16), in the form

$$\sum_k (k - \bar{k})^2 P(k) = |n - 2| P. \tag{23}$$

Substituting (23) and (1) into (16) we find the value of the dispersion:

$$\Delta^2_{me\,sq} = T\,|f_{co} - 2f|\,P. \tag{24}$$

Averaging over the range Δf, we find

$$\overline{\Delta^2_{me\,sq}} = \frac{1}{\Delta f}\int_{f_1}^{f_2} \Delta^2_{me\,sq}\,df. \tag{25}$$

Integrating, and dividing the result by $T^2\Delta f^2$, we obtain the square of the reduced mean-square error (17):

$$\delta^2_{me\,sq} = \frac{2P}{T\Delta f^2}\left|\left(\frac{f_{rp}}{2} - f_1\right)^2 \pm \left(f_2 - \frac{f_{co}}{2}\right)^2\right|. \tag{26}$$

If the utilized frequency range Δf is symmetrical about $f_{co}/2$, then

$$\delta^2_{me\,sq} = \frac{P}{T\Delta f}. \tag{27}$$

3. Some Conclusions

Besides the determination of errors, the obtained results aid in the evaluation and selection of a frequency range for pulse-frequency telemetering. In accordance with Equations (21) and (26), δ_{av} and $\delta_{me.\,sq.}$ depend on the width of the range Δf as well as on its position in the band 0 to f_{co} of the low-pass filter. In Figs. 3 and 4, the ratio of the middle frequency f_0, in the range Δf, to the cut-off frequency is plotted along the abscissae, while the values of the errors divided by multipliers, which are independent of Δf, from the right sides of the corresponding Expressions (21) and (26), are plotted along the ordinates. Curves are plotted for several values of Δf, expressed in fractions of f_{co}, and these characterize the movement of the range Δf in the band 0 to f_{co}. From Fig. 3 it can be seen that the mean error has a minimum value when the range Δf is symmetrical about $f_{co}/2$. Figure 4 shows the same for the mean-square error. Moreover, from Fig. 4 we establish the fact that the mean-square error diminishes as Δf approaches f_{co}.

To clarify the dependence of the noise-stability on the value of the range Δf, for a constant relationship between Δf and the filter's pass-band, it is necessary to analyze Equations (21) and (26), keeping in mind that P depends on the filter's pass-band (see Appendix).

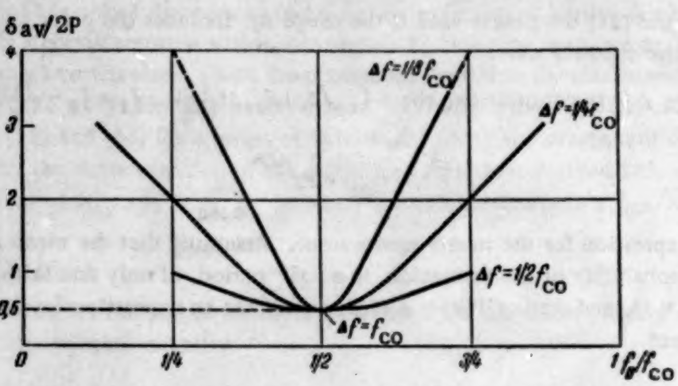

Fig. 3. Variation of the mean error with displacement of the range Δf in the frequency band 0 to f_{CO}.

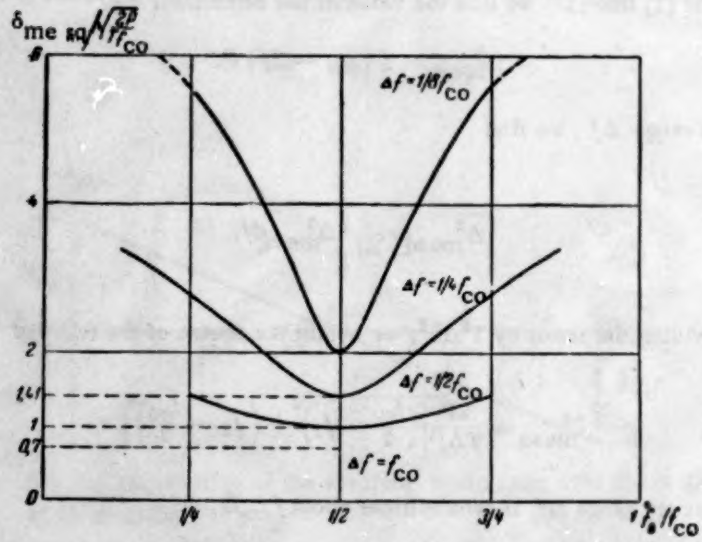

Fig. 4. Variation of the mean-square error with displacement of the range Δf in the frequency band 0 to f_{CO}.

From the analysis of Equation (26) for real values of noise ($P < 0.1$), it follows that, with increase in Δf, and a proportional increase in the band of frequencies in the channel, the ratio $P/\Delta f$ increases, i.e., the numerator increases more rapidly, and therefore, the value of the mean-square error increases. An examination of Equation (22) yields an analogous result for the mean error.

In this way, we can come to the following conclusions.

1. It is expedient to locate the range Δf as symmetrically as possible about $f_{CO}/2$.

2. If the range Δf and the practically arbitrary band of frequencies are connected thru some constant relationship, then, the narrower the range (or band) of frequencies, the greater is the noise-stability. This last condition opposes the condition for accuracy of telemetered transformations, i.e., the transformation of the measured parameter into a pulse repetition frequency and the reverse transformation at the receiving end.

APPENDIX

Comments Concerning the Initial Conditions of the Analysis

1. It was assumed above that the probability $P = P_p = P_s$ is known. For practical calculations it is necessary to indicate a method for finding P or P_p and P_s. To find these probabilities, it is necessary to know the laws

governing the noise distribution in the low-frequency channel (after the demodulator).

a) Let us first examine the case when the fluctuating noise has a normal distribution at the input to the receiver, containing a low-pass filter, a bilateral clipper and a pulse frequency meter.* There is equal probability P for the formation of unit distortion in a pulse or in a space if the mark-space ratio of the pulses is equal to two and $U_0 = U_C/2$.

If the effective value of the noise voltage U_N in the filter pass-band, and the level of the bilateral clipping U_0 are known, then the probability P is found from the well-known formula

$$P = P\{U_N > U_0\} = \frac{1}{2} - \Phi\left(\frac{U_0}{U_N}\right), \tag{28}$$

where Φ is the symbol for the Laplace integral,

$$\Phi(z) = \frac{1}{\sqrt{2\pi}}\int_0^z e^{-\frac{z^2}{2}}\,dz. \tag{29}$$

The values of this integral are given in tables.

b) Let us examine a receiver containing a band filter, a linear amplitude detector and the other circuits inherent in a pulse-frequency telemetering receiver – a bilateral clipper and a frequency meter. A high frequency signal, keyed amplitude-wise by telemetry pulses, arrives at the input of the receiver.

If the noise has a normal distribution in the high frequency channel, then, after the linear detector, in the absence of a signal, the noise has a simple Rayleigh distribution. This is a condition for finding P_s. In the presence of signals, the joint distribution of a sinusoidal signal and noise is governed by a generalized Rayleigh distribution law, also called the Rice distribution. This is a condition for the determination of P_p [1].

In the case when the effective values of the noise voltage in the filter pass-band U_N, and the levels of the signal U_c and the bilateral clipping** are known, the probability P_s of unit distortion in a space is found from the formula

$$P_s = e^{-\frac{1}{2}\left(\frac{U_0}{U_N}\right)^2}. \tag{30}$$

The probability P_p of unit distortion in the transmitted pulse is expressed by the equation

$$P_p = \frac{1}{U_N^2}\int_0^{U_0} U e^{-\frac{U^2+U_c^2}{2U_N^2}} I_0\left(\frac{UU_c}{U_N^2}\right) dU, \tag{31}$$

where I_0 is a zero-order Bessel function.

Values of this integral are given in tables.***

*In practice, this can be, for example, in the direct transmission of pulses without remodulation. Moreover, the distribution of fluctuation noise at the output of a linear detector is sometimes considered to be approximately normal for large signal-to-noise ratios.

**If the level of the bilateral clipping during the transmission of a space is U_0, then, during the transmission of a pulse, this level is calculated as $U_c - U_0$.

***See, for example, the curve in [1] or [2] from which the probability P_p is found as a function of $\frac{(U_0 - U_c)}{U_N}$ for a series of values of $\frac{U_c}{U_N}$.

In the case considered here, one should use, in the equations in parts 1 and 2, the value of P equal to the arithmetic mean:

$$P = \frac{P_p + P_s}{2}. \tag{32}$$

2. One of the initial conditions of the analysis is the existence, in the "video-pulse" channel of the receiver, a low-frequency filter* with cut-off frequency f_{co}. The basic filter, limiting the frequency spectrum at the receiver input, is the sub-carrier, band filter with pass-band W. The interaction of noise and signal in this band appears as beats of different components of the signal with noise components, and the noise components between themselves, which form, after demodulation, a group of difference frequencies. These difference (low) frequencies pass together with the signal and distort the reception.

The signal does not contain frequence components higher than W/2, and the main part of the noise spectrum appears in this band. Consequently, in the calculation of the noise stability of a telemetry receiver, having, at its input, a band filter with a pass-band W, one can assume that

$$f_{co} = \frac{W}{2}. \tag{33}$$

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* This filter should not be confused with the receiver's output filter located after the frequency meter. In the latter case, the output filter (for example, and indicating or recording unit) has a strong effect on the mean-square error but does not change the mean error.

A SINGLE-CYCLE MAGNETIC SHIFT REGISTER

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This paper includes an analysis of the functioning, approximate calculations and design and experimental data on the simplest register circuit using one core per bit. The influence of variations in the length of the shifting pulses on register operation is investigated.

Magnetic shift registers are widely used in the hardware of computers, automation devices, etc. Many register circuits employing magnetic materials with rectangular hysteresis loops are known [1-7]. These circuits provide not only for transmission of binary-coded signals, but also serve as a basis for constructing logical and controlling circuits. These circuits are simple, economical, reliable and have a long working life.

Best known are the two-cycle registers, which require two cores and four diodes for each bit. With the proper choice of the turns ratio of the input and output windings in the connecting circuits, the number of diodes required can be reduced to two [9]. Two-cycle circuits are sufficiently stable in operation. Analyses of them can be found in works [2-4]. However, two-cycle circuits require two series of shifting pulses and a significant number of cores and diodes. Therefore, there is definite interest in single-cycle registers [7, 8], which require one core per bit and, correspondingly, one source of shifting pulses.

There is given below an analysis of the functioning of a single-cycle register which explains the essential physical processes occurring during register operation, and approximate computations are also given.

Principles of Register Operation

Figure 1 shows the simplest circuit of a single-stage register. Each cell (stage) consists of a transformer with three windings (input winding W_1, shifting winding W_2 and output winding W_3), a diode B, a condenser C, and resistor R.

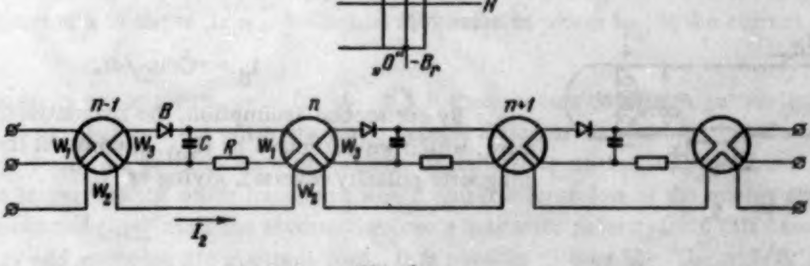

Fig. 1.

The condenser in the connecting link temporarily remembers the state of the core in the form of its charge. Discharge occurs across the input winding after cessation of the shifting pulse, which allows the use of just one core per bit.

It is necessary, for normal operation of the register, that when the n'th core is read, a signal be transmitted only to the following (n + 1)'st core. In the circuit given, this condition is met. We consider the transmission of a "1" from the n'th core to the (n + 1)'st. Let the "1" be realized physically by the core's residual induction, B_r, and let "0" be realized by $-B_r$. The shifting pulse changes the core's magnetic polarity from state "1" to state "0". A current pulse flows through the connecting link, charging the condenser. The resistor R limits the current which is taken off by the input winding of the (n + 1)'st core. The incomplete blocking of this current is not important, since in the (n + 1)'st core, at the same time, the shifting pulse induces a field opposed to the field of the writing current. Charging of the capacitor continues until the n'th core has had its magnetic polarity completely reversed. The shifting pulse must cease at this moment. The condenser begins to discharge across the input winding of the (n + 1)'st core, magnetizing it to state "1." With this there is induced, in the output winding of the (n + 1)'st core, an emf of such polarity that the diode prevents passage of current. The information stored in the n'th core is thus transmitted ahead only to the (n + 1)'st core.

The reverse motion of the code is limited, since the condenser in the feedback path is shunted by the forward impedance of the diode, and can not be charged sufficiently.

Analysis of Register Operation

It is not possible to take into account all the factors which affect the operation of a register. Therefore, a number of common assumptions are made in our analysis: 1) the core material has a rectangular hysteresis loop; 2) during the entire process of magnetic polarity reversal, the field acting on the core has a constant strength, H_m;* 3) the shifting current pulse has a rectangular shape; 4) the forward impedance of the diode during capacitor charging is constant, and equal to R_d; 5) the back impedance of the diode is small, and does not affect register operation; 6) the analysis does not take into account the induction, leakage or parasitic capacitance of the windings, or the capacitance of the diodes.

In what follows, the following notation will be used: Φ_r is the residual magnetic flux in webers, H is the magnetic field strength in ampere-turns/meter, U_{Cf} is the voltage on the capacitor at the moment when the shifting pulse terminates, t_c is the charging time of the capacitor in seconds, l is the mean length of the core's magnetic lines of force in meters, t_m is the time taken to reverse the core's magnetic polarity when the capacitor discharges in seconds, S is the area of the core's lateral section in meters^2, C is the capacity in farads, R is the resistance in the connecting link in ohms, R_d is the diode's forward resistance in ohms, t_d is the total time for discharging the condenser in seconds and i_d is the discharge current of the condenser in amperes.

The transmission of information from one core to the next may be decomposed into two individual steps: discharge and charging of the condenser.

Condenser discharge. The circuit for condenser discharge is given in Fig. 2. With a "1" read from the previous core, the condenser is charged up to the voltage U_{Cf} at the moment the shifting pulse terminates and all the register's cores have gone to state "0." The discharge current i_d induces a change in the core's magnetic field. By Kirchhoff's law, $u_C = W_1 d\Phi/dt + i_d R$.

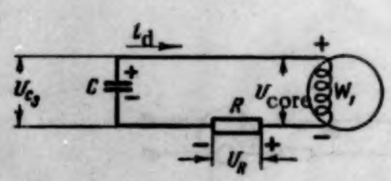

Fig. 2.

The discharge current of the condenser is determined from the relationship:

$$i_d = -C du_C/dt.$$

By our second assumption, the magnetizing field acting on the core, which we designate by H_{m1}, is constant throughout the period of magnetic polarity reversal, giving us

*H_m is the material's dynamic characteristic corresponding, in the static regimen, to the coercive force. H_m is larger than the coercive force since it includes the effects of eddy currents and magnetic viscosity. Methods of measuring H_m as a function of the magnetization time are described in the literature (Cf., e.g., [4]).

$$i_d = \frac{H_{m1} l}{W_1} = \text{const} \tag{1}$$

The voltage on the condenser is

$$u_C = U_{Cf} - \frac{i_d t}{C}. \tag{2}$$

If we take into account that, by Kirchhoff's law, $\Phi(t) = \Phi_r$ for t = 0 then, from (1) and (2), we get

$$\Phi(t) = \frac{U_{C_f} - i_d R}{W_1} t - \frac{i_d t^2}{2CW_1} - \Phi_r. \tag{3}$$

Figure 3 gives the graphs of $\Phi(t)$ for various values of U_{Cf}. The process of magnetic polarity reversal terminates at the moment t_m, when the magnetic flux in the core becomes equal to Φ_r, i.e., $\Phi(t_m) = \Phi_r$.

By solving Equation (3) with respect to t_m, we obtain

$$t_m = \frac{C(U_{C_f} - i_d R)}{i_d} - \sqrt{\left[\frac{C(U_{C_f} - i_d R)}{i_d}\right]^2 - \frac{4\Phi_r CW_1}{i_d}}.$$

For complete reversal of the core's magnetic polarity, it is necessary that the following inequality hold:

$$\left[\frac{C(U_{C_f} - i_d R)}{i_d}\right]^2 \geqslant \frac{4\Phi_r CW_1}{i_d} \quad \text{or} \quad U_{C_0} \geqslant 2\sqrt{\frac{H_{m1} l \Phi_r}{C}} + \frac{H_{m1} lR}{W_1}.$$

We find the least value of voltage on the condenser necessary for complete reversal of the core's magnetic polarity from the formula:

$$U_{C_0} = 2\sqrt{\frac{H_{m1} l \Phi_r}{C}} + \frac{H_{m1} l}{W_1} R. \tag{4}$$

With this, the time of magnetic polarity reversal equals

$$t_m = \frac{C(U_{C_0} - i_d R)}{i_d} = \frac{CU_{C_0} W_1}{H_{m1} l} - RC. \tag{5}$$

If $U_{C_f} < U_{C_0}$, the polarity reversal will be partial.

If a "0" is read from the previous core then, at the moment when the shifting pulse terminates, a noise voltage U_n will appear on the condenser. If, with this, the discharge current from the condenser induces in the input winding a field less than the coercive force, the flux in the succeeding core will not vary and, consequently, the noise will not accumulate in the register. With this, $U_{C_n} = i_{H_C} R$, where i_{H_C} is the current corresponding to the coercive force.

However, as shown by experiments, even for $U_{C_n} = i_d R$ there occurs no significant variation of the magnetic flux. This is related to the fact that the noise discharge current differs little from i_C, but the time of its action on the core is limited. From the point of view of stability of register operation, it is desirable that the admissible value of U_{C_n}/U_{C_0} be large. On the other hand, this would entail a large loss of the energy stored in the condenser when a "1" is transmitted (U_{C_n} must not reverse the core's magnetic polarity). In this case, the requirements of operational stability and economy are contradictory. It is possible to take $U_{C_n}/U_{C_0} = 1/3$. This magnitude is probably close to the optimum since, in this case, the energy loss will be about 10% of what is initially stored on

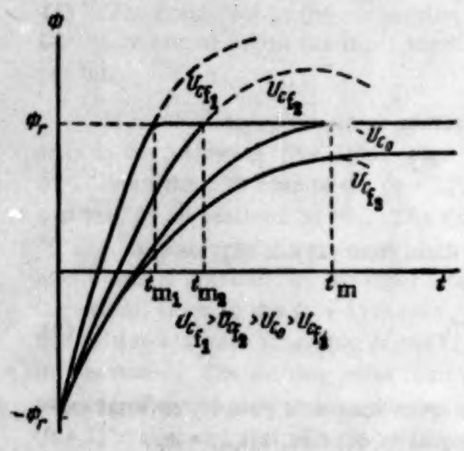

Fig. 3. Temporal variation of flux as condenser discharges.

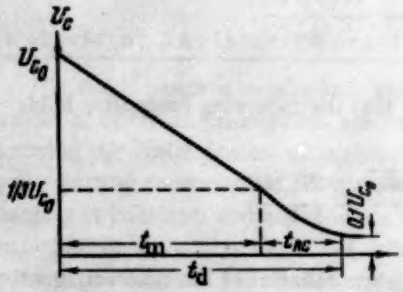

Fig. 4. Voltage variation on the condenser during discharge.

the condenser, and the noise voltage capable of being engendered in the register may reach a value of one third the signal voltage, which would not disrupt register operation. Taking $U_{C_n}/U_{C_0} = 1/3$, we obtain from (1), (2) and (4) that

$$U_{C_n} = 3i_d R = \frac{3H_{m1}}{W_1} R = 3\sqrt{\frac{H_{m1}\Phi_r}{C}}. \tag{6}$$

Whence,

$$W_1 = \frac{H_{m1} R}{\sqrt{\frac{H_{m1}\Phi_r}{C}}}.$$

If the number of turns is chosen by this formula, then the time of magnetic polarity reversal is found, from (5) and (6), to be

$$t_m = 2RC.$$

After magnetic polarity reversal, the input winding will have low impedance and discharge of the capacitor will occur exponentially, with time constant RC (Fig. 4). It is possible to consider that the condenser is completely discharged when the voltage on it reaches $0.1\,U_{C_0}$, i.e., $(1/3)U_{C_0}\exp(-t_{RC}/RC) = 0.1\,U_{C_0}$. It follows that $t_{RC} = 0.8RC$, and the total discharge time equals $t_d = t_m + t_{RC} = 2.8RC$.

Condenser Charging. We now consider the most difficult operating case, when a "1" is simultaneously read from all cores of the register. With this, in the input and output windings of the n'th core there will be a flow of current opposing its magnetic polarity reversal and affected by the simultaneous polarity reversals of the preceding and succeeding cores (Fig. 5).

The average value of the resulting field strength equals the algebraic sum of the field strengths acting on the core:

$$H_{m2} = H_2 - H_1 - H_3 = \frac{I_2 W_2}{l} - \frac{i_1 W_1}{l} - \frac{i_3 W_3}{l}.$$

It follows from Fig. 5 that

$$i_3 = i_c + i_1 = \frac{c\,du_C}{dt} + i_1.$$

We find, from the last relationships, that

$$i_3 = \frac{CW_1}{W_3(1 + W_1/W_3)}\frac{du_C}{dt} + \frac{(H - H_{m2})\,l}{W_3(1 + W_1/W_3)},$$

$$i_1 = \frac{(H_2 - H_{m2})\,l}{W_3(1 + W_1/W_3)} - \frac{C}{1 + W_1/W_3}\frac{du_C}{dt}. \tag{7}$$

By Kirchhoff's law,

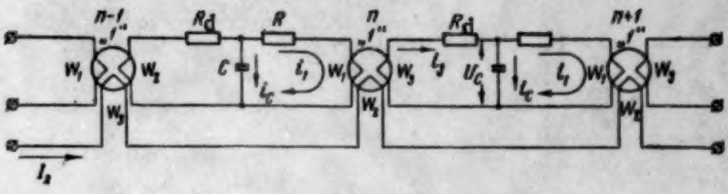

Fig. 5. Condenser charging circuit.

$$-\frac{d\Phi}{dt}W_3 = i_3R_d + u_C \text{ and } -\frac{d\Phi}{dt}W_1 = i_1R - u_C.$$

By substituting in these equations the values for the currents i_1 and i_3, we obtain the equation

$$u_C + \frac{C\,[R + R_d\,(W_1/W_3)^2]}{(1 + W_1/W_3)^2}\,\frac{du_C}{dt} = \frac{(H_s - H_{ms})\,l\,[R - R_dW_1/W_3]}{W_3\,(1 + W_1/W_3)^2}$$

the solution of which has the form:

$$u_C = E\left[1 - \exp\left(-\frac{t}{\tau}\right)\right]; \tag{8}$$

where

$$E = \frac{(H_s - H_{ms})\,l\,(R - R_dW_1/W_3)}{W_3\,(1 + W_1/W_3)^2} \text{ and } \tau = \frac{C\,[R + R_d\,(W_1/W_3)^2]}{(1 + W_1/W_3)^2} \tag{9}$$

are defined by the boundary conditions that $u_C = 0$ for $t = 0$ and $du_C/dt = 0$ for $t = \infty$.

The expressions for the currents i_3 and i_1 can be rewritten in the following form:

$$i_3 = \frac{CW_1/W_3}{1 + W_1/W_3}\,\frac{E}{\tau}\exp\left(-\frac{t}{\tau}\right) + \frac{E\,(1 + W_1/W_3)}{R - R_dW_1/W_3},$$
$$i_1 = \frac{E\,(1 + W_1/W_3)}{R - R_dW_1/W_3} - \frac{C}{1 + W_1/W_3}\,\frac{E}{\tau}\exp\left(-\frac{t}{\tau}\right).$$

It is possible, knowing the laws by which the currents i_3 and i_1 vary in the connecting link and by which the voltage u_C varies on the condenser, to determine the law defining the velocity with which the magnetic flux in the core varies during charging of the condenser:

$$-\frac{d\Phi}{dt} = \left[\frac{CR_d\,(W_1/W_3)\,E}{W_3\,(1 + W_1/W_3)\,\tau} - \frac{E}{W_3}\right]\exp\left(-\frac{t}{\tau}\right) + \frac{E\,(R_d + R)}{RW_3 - R_dW_1}.$$

We now determine the time of condenser charging t_c during which the core changes state, from Φ_r to $-\Phi_r$. At the instant $t = t_c$, $\Phi(t) = -\Phi_r$ and therefore, we obtain from the previous expression that

$$2\Phi_r = \frac{E\,(R_d + R)\,t_c}{RW_3 - R_dW_1} - E\left[\frac{CR_dW_1/W_3}{W_3\,(1 + W_1/W_3)} - \frac{\tau}{W_3}\right]\left[1 - \exp\left(-\frac{t_c}{\tau}\right)\right]$$

The equation just obtained was not solved explicitly for t_c. But, from this equation and from (8), we obtain the ratio:

$$\frac{t_c/\tau}{1 - \exp\left(-\frac{t_c}{\tau}\right)} = \frac{R - R_dW_1/W_3}{R_d + R}\left[\frac{2\Phi_rW_3}{U_{C_s}\tau} + 1 - \frac{CR_dW_1/W_3}{(1 + W_1/W_3)\,\tau}\right]. \tag{10}$$

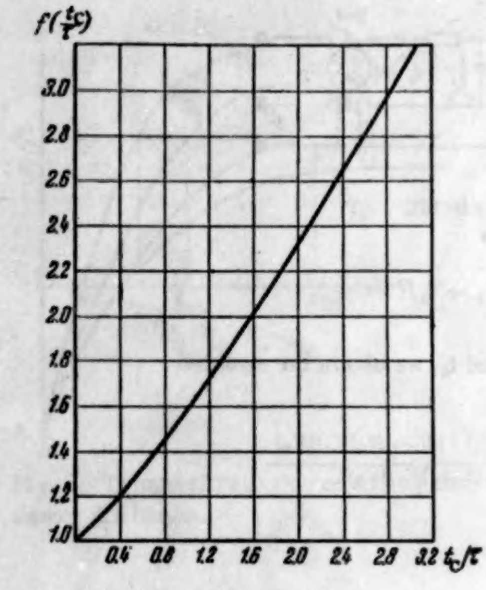

Fig. 6.

The right side of Equation (10) can be computed. By using the graph of $(t_c/\tau)/(1-\exp(-t_c/\tau)) = f(t_c/\tau)$ (Fig. 6), it is possible to determine t_c/τ and, by computing τ from Formula (9), to determine the capacitor charging time t_c.

We return to Expression (8). For a constant magnitude of shifting pulse current, the voltage U_{Cf} has a maximum value for some optimal ratio W_1/W_3. With this, the ampere-turns I_3W_3, necessary for charging the condenser to U_{Cf} will be a minimum.

To find the optimal value of W_1/W_3 analytically is quite difficult. An approximately optimal turns ratio may be found by determining the maximum of the function $E = f(W_3)$ for a given W_1:

$$W_{3\,opt} = W_1(1 + 2R_d/R)$$

or

$$(W_1/W_3)_{opt} = R/(R + 2R_d). \tag{11}$$

An exact value of $(W_1/W_3)_{opt}$ can be found by a series of tentative calculations.

<u>Reverse Signal Transmission.</u> When a "1" is read from the n'th core, a voltage is induced in its input winding. As a result, there occurs a decrease of the condenser's charge (if a "1" was also read from the (n −1)'st core) or an overcharge of the condenser (if an isolated "1" were transmitted). In the latter case, after termination of the shifting pulse, there occurs a partial variation in the flux in the (n −1)'st core, which leads to an increase of noise at the following step. It has been shown experimentally that no further increase of noise occurs in the register, if the overcharge voltage, U_{C_1}, when an isolated "1" is read, is some five to ten times less than the voltage U_{C_3} (Fig. 7).

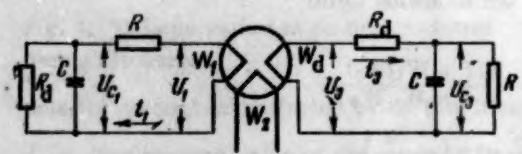

Fig. 7. For computing reverse signal transmission.

We now consider that R must be so that this last condition holds.

The voltages U_{C_1} and U_{C_3} can be defined by means of Duhamel integrals:

$$U_{C_1}(t) = h_1(0)\,u_1(t) + \int_0^t h_1'(t-\lambda)\,u_1(\lambda)\,d\lambda,$$

$$U_{C_3}(t) = h_3(0)\,u_3(t) + \int_0^t h_3'(t-\lambda)\,u_3(\lambda)\,d\lambda,$$

where $h_1(t) = (1/(1 + R/R_d))\,(1 - \exp(-t/\tau))$ and $h(t)_3 = (1/(1 + R_d/R))\,(1 - \exp(-t/\tau))$ are transfer functions, and $\tau = CR/(1 + R/R_d)$.

The voltages $u_1(t)$ and $u_3(t)$ are related by the expression, $u_1(t) = u_3(t)\,W_1/W_3$.

As the result, we get

$$\frac{U_{C_3}}{U_{C_1}} = \frac{\dfrac{1}{1 + R_d/R}\dfrac{W_3}{W_1}\displaystyle\int_0^t \frac{\exp\left(-\frac{t-\lambda}{\tau}\right)}{\tau}\,u_1(\lambda)\,d\lambda}{\dfrac{1}{1 + R/R_d}\displaystyle\int_0^t \frac{\exp\left(-\frac{t-\lambda}{\tau}\right)}{\tau}\,u_1(\lambda)\,d\lambda} = \alpha.$$

By taking (11) into account, we find the desired formula: $R = (\alpha - 2)R_d$, where α is between 5 and 10.

Influence of the Length of the Shifting Pulse on Register Operation

We now consider the case, of practical interest when $I_2 > I_{20}$ and $t_2 > t'_C$, where I_{20} is the minimum shifting current I_2, for which, during time t_{c0}, the condenser is charged up to U_{C_0}, and t_c' is the time taken to reverse the core's magnetic polarity by current I_2.

With this, there is complete read-out from the n'th core. At the moment of time t'_c, $U_C > U_{C_0}$. During the time $\Delta t = t_2 - t'_C$, the condenser partially discharges into the (n + 1)'st core, already shifted in magnetic polarity. Two variations are possible:

a) during the interval Δt, the condenser discharges to $U_C < U_{C_0}$, and there is an attenuation of information in the register;

b) during the interval Δt, the condenser discharges to $U_C \geq U_{C_0}$ (Fig. 8), with which normal operation of the register is assured.

In order that $u_C(t_2) = U_{C_0}$, it is necessary that

$$U'_{C_f} = U_{C_0} \exp \frac{\Delta t}{RC}, \tag{12}$$

where U'_{C_f} is the voltage on the condenser at the instant t'_C.

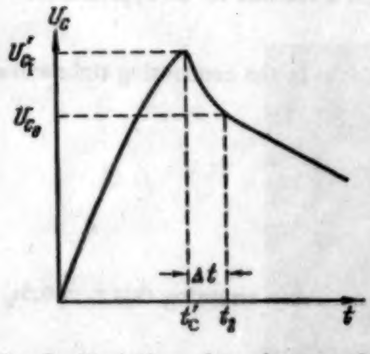

Fig. 8. Variation of condenser voltage in a forcing regimen.

We have, from Formula (8),

$$U'_{C_f} = E\left[1 - \exp\left(-\frac{t'_c}{\tau}\right)\right]. \tag{13}$$

It follows from Formula (9) that $E = A(H_2 - H_m)$, where

$$A = \frac{l\,(R - R_d W_1 / W_2)}{W_2\,(1 + W_1 / W_2)^2}, \quad H_2 = I_2 W_2 / l$$

and H_m can be taken off the graph of $H_m = f(t)$ for the instant t'_C.

It follows from (12) and (13) that

$$I_2 = \frac{H_m l}{W_2} + \frac{U_{C_0} \exp \dfrac{\Delta t}{RC}\, l}{AW_2\left[1 - \exp\dfrac{(t_2 - \Delta t)}{\tau}\right]} \tag{14}$$

By assuming that $\Delta t \ll t_2$, and using the notation

$$\gamma = \frac{U_{C_0}\, l}{AW_2^4\left[1 - \exp\left(-\frac{t_2}{\tau}\right)\right]},$$

we obtain

$$I_2 = \frac{H_m l}{W_2} + \gamma \exp \frac{\Delta t}{RC}.$$

Thus, a variation of the shifting pulse length by Δt requires an increase in the current I_2. During the interval Δt, the condenser ineffectively discharges, with time constant RC, through the writing winding of the (n + 1)'st core, which is saturated by the current I_2 during this time.

To remove this flaw, more complex circuits are used, with LC delays in the connecting links (Fig. 9) or with subsidiary diodes, B_2 (Fig. 10).

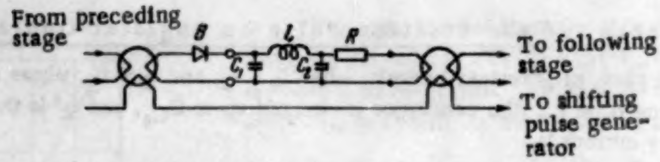

Fig. 9. Circuit for a one-cycle register with an LC lag in the connecting link.

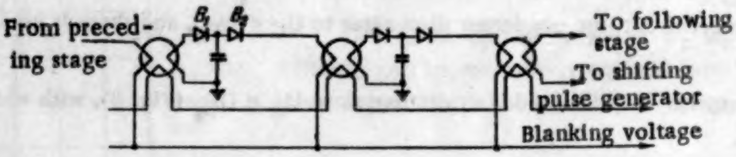

Fig. 10. Circuit for a one-cycle register with a subsidiary diode B_2 in the connecting link.

In the latter case, a blanking voltage is applied during the time the shifting pulse is applied.*

Register Design

On the basis of the circuit analysis given above, it is possible to present a method for an approximate register design.

Assumed as given are the material and dimensions of the cores, the diodes in the connecting links and also the speed of action.

The sequence of design steps follows.

1. The period between shifting pulses is $T_1 = 1/f_1$.

2. R and C are determined from the following formulae:

a) R is between $5R_d$ and $10R_d$, b) $t_d + t_c \leq T_1$, where $t_d = 2.8RC$. Tentatively assuming that $t_c = 0.5t_d$, we get that $R \leq T_1/4.2C$.

3. The time to reverse the core's magnetic polarity upon condenser discharge is determined from the formula $t_m = 2RC$.

4. From the graph of $H_m = f(t)$, H_{ml} is determined for $t = t_m$.

5. The residual magnetic flux is $\Phi_r = B_r S$.

6. The voltage on the condenser necessary for complete reversal of the core's magnetic polarity is $U_{C_0} = 3\sqrt{H_{ml} l \Phi_r / C}$.

7. The number of turns on the input winding is $W_1 = H_{ml} LR / \sqrt{H_{ml} l \Phi_r / C}$.

8. The number of turns on the output winding must satisfy the following inequality: $W_2 \geq W_1 (1 + 2R_d/R)$.

9. The time constant of condenser charging is $\tau = C(R + R_d (W_1/W_2)^2)/(1 + W_1/W_2)^2$.

10. We compute the function:

$$\frac{t_c}{\tau\left[1-\exp\left(-\frac{t_c}{\tau}\right)\right]} = \frac{R - R_d W_1/W_2}{R + R_d}\left[\frac{2\Phi_r W_2}{U_{C_0}\tau} + 1 - \frac{R_d C W_1/W_2}{(1 + W_1/W_2)\tau}\right].$$

***The circuit with the blanking voltage was first presented by V. A. Zhozhikashvili and K. G. Mitiushkin.**

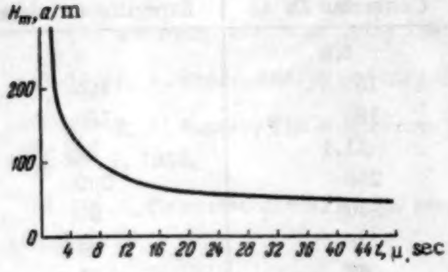

Fig. 11.

11. From the graph (Fig. 6, we determine $a = t_c/\tau$.

12. The time for charging the condenser is given by $t_c = \tau a$.

13. We next determine the maximum speed of operation of the register, characterized by the quantity $f_2 = 1/T_2 = 1/(2.8RC + t_c)$. f_2 must be greater than, or equal to, f_1. Otherwise, it is necessary to decrease RC (Cf. step 2).

14. $$E = \frac{U_C}{1 - \exp\left(-\frac{t_c}{\tau}\right)}.$$

15. From the graph of $H_m = f(t)$, H_{m2} is determined for $t = t_c$.

16. $$H_2 l = I_2 W_2 = \frac{E\,(1 + W_1/W_3)^2 W_3}{R - R_d W_1/W_3} + H_{m2} l.$$

17. The pulse power during one cycle is $P_p = U_2 I_2 = 2\Phi_r W_2 I_2/t_c$.

18. The average power in one cycle is $P_{av} = P_p t_c/T_2$.

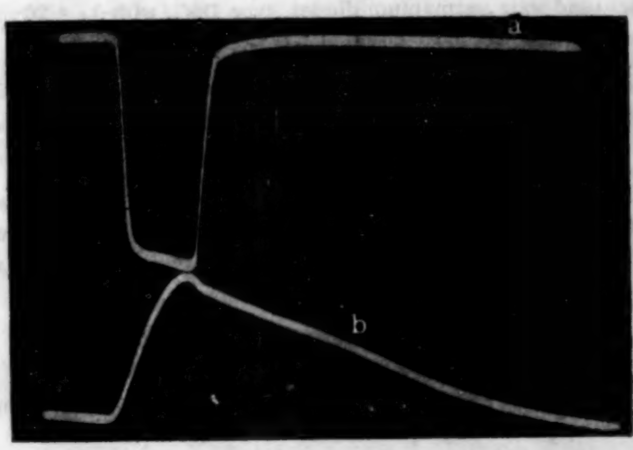

Fig. 12. a) is the shifting pulse current I_2 and b) is the voltage on the condenser.

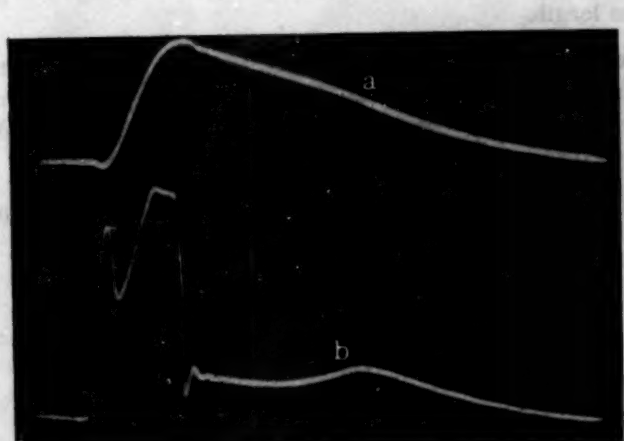

Fig. 13. a) is the voltage on the condenser and b) is the current in the input winding.

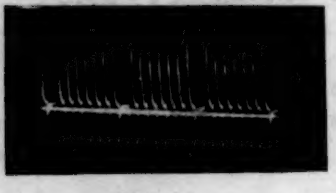

Fig. 14.

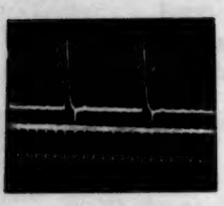

Fig. 15.

Electrical Parameters of the Register

Parameter	Computed Data	Experimental data
U_{C_0}, volts	6.5	5.2
W_1	155	155
W_3	185	185
t_c, microsecond	21.4	19
W_2	200	200
I_2, milliamps	100	90
t_m, microsecond	48	45
t_d, microsecond	67	71
P_p, watts	0.625	0.51
P_{av} watts	0.134	0.1
Speed in kilocycles	10	10*

*The maximum speed equals 11 kilocycles.

In order to test the various assertions of the analysis and the computational method of design, an eight-digit register was designed and built.

The core elements of the register took the form of annular rings, 9 mm in diameter and 3 mm high. The core material was alloy 50 NP of a 20 μ strip thickness. The strip had four turns. The function $H_m = f(t)$ is given on Fig. 11. The diodes used were germanium diodes, type D9G, with R_d = 20 ohms. The computed speed of operation was f = 10 kilocycles.

For the generator of the shifting pulses, a 6P6 vacuum tube was used. The register worked in a ring circuit. It follows from the Table which gives the register's electrical parameters that the experimental and computed data coincide.

Figures 12-15 show photographs of the voltage and current, illustrating the character of the basic physical processes during register operation. One photo (Fig. 13) confirms quite well one of the basic assumptions of the analysis and computations: the constancy of the condenser's discharge current during magnetic switching of the (n + 1)'st core.

SUMMARY

The analysis given of the simplest circuit for a one-cycle register permits its approximate design with an accuracy which is sufficient for practical purposes.

The design method given can be generalized to the reading from two and more cores.

It was shown that, in contradistinction to two-cycle registers, the one-cycle register is critically affected by variations in shifting pulse length.

The reverse signal transmission occurring in the circuit, with its attendant shunting of the current to the (n + 1)'st core, when a "1" is read from the n'th core, leads to a dependency on the structure of information in the register and of the form of the signal on the condenser. This might create difficulties in constructing logical circuits with single-cycle registers.

The simplest circuit, considered here, can be recommended for use in ring counter circuits and in registers which are not too long.

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SCIENTIFIC SEMINAR ON PNEUMO-HYDRAULIC AUTOMATA

From March 17, to March 19, 1959, at the Institute of Automation and Remote Control of the AN SSSR (Academy of Science of the USSR), there was held the Second All-Union Seminar-Meeting on Pneumo-Hydraulic Automata, organized in 1957.

In attendance were 328 people from 63 organizations, 62 of the individuals representing organizations working in the domains of pneumo- and hydro-automata in Leningrad, Baku, Kiev, Sverdlovsk, Khar'kov, Cheliabinsk, Dnepropetrovsk and many other cities.

Taking part in the work of the seminar were specialists in questions of pneumo- and hydro-automata from other national democratic (Soviet block) countries: Prof. Lu Iuan'-tsin (KNR) (Chinese National Republic), Dzhozef Kveton and Ian Khampl (Czechoslovakian National Republic) and Britall and Maier (GDR)(German Democratic Republic) and others.

In opening the session, seminar leader Prof. M. A. Aizerman, presented the problems on which the seminar would work.

First heard was the paper of Engineer V. A. Nikitin (Giprogaztopprom, Moscow), "The AUS pneumatic assembly system—the basic automated complex in oil-refining." The paper contained brief characterizations of the technological means for automatic inspection, regulation and control which are necessary for the automation of complex oil-refining processes, and also provided results of the experimental use of the AUS system.

R. A. Auzan (TsNIIKA, Moscow), in her paper on the subject, "An investigation of the dynamic characteristics of the pneumatic regulators of the AUS system," presented the design methodology and results of an experimental investigation, of the regulating blocks of the AUS system.

Next was read the paper of Engineer Ferner (GDR), "Pneumatic regulators and computing links." The paper described analog pneumatic elements, developed in the GDR, which execute the computational operations which might be required for solving regulation problems. The system works at low pressures (0 to 100 mm of water), which only in extreme cases of necessity is increased to the range 0.2 to 1.0 atmospheres. Also considered in the paper was the superiority of a low-pressure system in comparison with those of the higher pressures which are usually employed.

V. Britall (GDR) answered the questions raised by Ferner's paper.

The evening session was opened by V. V. Volgin (Moscow Energetics Institute) with the paper, "Pneumohydraulic methods of implementing advancing." The paper presented results of a theoretical investigation related to the use of pneumatic blocks for forward and reverse advancing in the regulation of technological processes by pneumatic regulators. The stability of single-loop automatic control systems containing advancing blocks was considered, and certain conclusions were given as to the influence of forward and reverse advancing on the quality of the transient response.

L. L. Feigel'son (KB Tsvetmetavtomatika, Moscow) presented the communication, "A brief survey of the designs of pneumatic transducers in domestic and foreign production," which was devoted to an analysis of the principal circuits and designs of pneumatic pressure transducers for the discharging and transmission of pressure. The designs of five domestic organizations and 23 foreign companies were considered.

In his communication, entitled "The use of an extremum-regulator for the inspection and control of a reaction by certain chemical processes using the thermal effect," V. M. Dobkin (Moscow) related the work which he had carried out in conjunction with M. L. Kurskaia and Iu. O. Ostrovskii (IAT). In this work were considered variations of a scheme for regulating a process with two interacting components by means of an extremum-

regulator, and also a scheme for automatic thermotitration for determining the content of one of the products in the reactive mass. The results of experiments performed with laboratory equipment were given.

E. M. Nadzhafov (IAT AN SSSR) in his paper on work done in collaboration with Iu. I. Ivlichevoi (IAT AN SSSR, Moscow) reported on the results of work in creating pneumatic universal multiplying-dividing devices, devices for extracting square roots and membrane-less devices for adding continuously-varying pneumatic signals in the range from 0 to 1.0 atmospheres.

A. A. Tal' (IAT AN SSSR, Moscow), in the paper, "Pneumorelays and their applications in pneumatic discrete computing devices," (work done jointly with T. K. Berends and N. V. Grishko) showed that discrete computing devices could be made on the basis of pneumatic apparatus. It was shown that the basic element of this apparatus is the universal pneumorelay, the circuit and miniaturized dimensions of which were developed in the IAT. The properties of this pneumorelay are such that it can be used for synthesizing all possible discrete pneumatic computing devices. The circuits of four devices (a generator, a separator, a pulse counter and an interval indicator for a continuously-varying quantity), constructed of pneumorelays, were given. It was shown that it is possible to use pneumatic interval indicators in devices for the automatic load change-over of pneumatic regulators.

L. A. Zalmanzon (IAT AN SSSR, Moscow), in the paper, "Aerodynamic oscillation generator (sine-wave generator)," reported on the development of a new type of oscillation generator with no moving parts, excitation of oscillation occurring only as a result of the interaction of air currents. Results of a theoretical and an experimental investigation of generator characteristics were reported, and the possibilities were cited of its use for obtaining the frequency characteristics of regulators and regulated objects, for driving various vibration equipments and in the solution of a host of other technical problems.

L. A. Zalmanzon also reported on the development, done jointly with A. I. Semikovoi (IAT AN SSSR, Moscow), of a multiplying-dividing device, constructed of nozzle-duct type elements, designed to function in pneumoautomata devices. With the introduction of an additional feedback path, this device can also implement the function of extracting square roots.

V. N. Dmitriev (IAT AN SSSR, Moscow) reported on the piston follower pneumodrive, developed in the IAT AN SSSR, cited its basic parameters and gave its empirical static and dynamic characteristics. It was noted that in the drive distributed vents were used instead of valves.

In the paper of E. V. Gerts (Institut Mashinovedemiia, AN SSSR, Moscow), "An investigation of resino-webbing membrane power drives," there were considered the existing formulae for determining the working stress or effective membrane area, and data were given from investigations made by the paper's author. A comparison of experimental and theoretical (computed) data was made .

The morning session of March 18 began with the report by A. I. Semikovoi of work done jointly with L. A. Zalmanzon (IAT AN SSSR, Moscow) in the investigation of the characteristics of pneumatic chambers used as adders. The report gave data from an investigation into the influence of the compressibility of air on the accuracy of addition by means of pneumatic chambers, and ways were cited for increasing the accuracy of pneumatic chamber-adders.

In the paper of the chief designer of the "Regula-vivoi," Dzhozef Kveton (Czechoslovakian National Republic), there was given a list of the pneumatic instruments released and recently developed by the factory. Two groups of pneumatic instruments have appeared: a basic group for the range of air pressure from 0.2 to 1.0 atmospheres, and systems for controlling boilers, working in the pressure range from 0.7 to 3.5 atmospheres.

The basic system was developed in the form of assemblies; right now, regulatory blocks with motion compensation are being produced, and next year production will begin on regulatory blocks with power compensation. The speaker also reported on the executive mechanisms (membranes and pistons) produced in the factory, and on the work on the creation of new types of transducers.

Dzhozef Kveton answered numerous questions from the conference participants.

P. E. Baloban (VTI, Moscow), in his report on work done jointly with G. N. Makhan'kov, "Unified hydraulic regulating devices," gave data on unified hydraulic regulators which work reliably when ordinary municipal water is used as the working agent. G. N. Makhan'kov (VTI, Moscow) provided additional information on experimental regulators which work on aqueduct waster and on turbine condensate.

V. Britall (GDR) gave a paper on the subject, "Hydraulic and composite hydraulic regulatory systems in the GDR."

Two types of hydraulic regulators are produced in the GDR: regulators constructed on the assembly principle, for which systems of electrical remote task assignment have been developed, and normal hydraulic regulators with flow piping. Work is being done in combining hydraulic systems with low-pressure pneumatic systems.

A number of measuring devices for hydraulic regulators have been developed (for measuring drops under high static pressures, for working under the conditions of a viscous or agressive medium, etc.)

A. M. Duel' (Khar'kov), in his paper on "Certain questions as to the use of hydraulic regulatory apparatus in complex schemes of automated power plants," considered the possibilities of using hydraulic auto-regulators for complex automation of the basic plants of thermal power stations.

The evening session of March 18 opened with the paper, "Systems of coupled regulation," of V. N. Veller (All-Union Heat Technology Institute, Moscow). The paper included comparisons of the various designs for coupled regulation of steam turbines. The disadvantages and advantages of each of the designs treated were mentioned.

V. D. Mironov (VTI, Moscow), in the paper, "Electronic-hydraulic regulator, type EGP-T," spoke of the electronic-hydraulic regulator, based on an electronic regulation system and a unified series of hydraulic regulators, which was developed at VTI. All regulator elements have undergone lengthy tests under industrial conditions.

Ian Khampl (Czechoslovakian National Republic), in his presentation, spoke of the work of the "Krzhyk-Smikhov" factory in the production of electro-hydraulic regulators.

E. F. Alekseev, in the paper "On certain questions of the dynamics of rotating-piston drives as a result of experimental investigation," considered questions of the dynamics of electro-motor hydro-drives in relation to the characteristics of the former.

According to the speaker, it has been experimentally established that oscillations in pressure and rotational speed of the output shaft, which arise under certain conditions, depend on particular design features on the machines and on the magnitude of the load on the drive.

In the paper of V. A. Khokhlov (IAT AN SSSR, Moscow), "On one method of calculating the inertial load of hydraulic executive mechanisms with trottle control, by an analysis of servo system dynamics," a method was presented for analyzing the quality of the transient responses in hydraulic and electro-hydraulic throttle-control servo systems. It wasshown that it is convenient, in engineering practice, to represent by an equivalent lag the inertial load connected with the input shaft of the hydro-motor in the analysis of servo system dynamics. A formula was provided for determining the magnitude of this lag.

V. I. Gusakov gave a paper on the subject, "Increasing the speed of action of a throttle hydro-drive by stabilizing the pressure at the throttle aperture." The speaker presented various methods of stabilizing the pressure in hydrosystems.

In the report of V. P. Temnogo (IAT AN SSSR, Moscow), "Methods of increasing the gain of industrial hydraulic servos," claims were made as to the efficacity of using hydraulic servo drives, both for the control of unitary processes and for complex automation, and an analysis of the basic parameters determining drive gain was provided. Six schemes of hydraulic servo drives were compared.

V. M. Dvoretskii (IAT AN SSSR, Moscow), in his paper, "Hydraulic advancing blocks for general industrial use," spoke of the principles of action and the characteristics of hydraulic advancing blocks of the compensated type. Data on their experimental investigation were given.

In the opening session on March 19 B. L. Korobochkin (the Ordzhonikidze Stankozavod, Moscow), in his paper, "The automatic control of hydraulic transformer stands," spoke on work being done in creating new types of drives for the chief movement of stands – hydrodynamic transformers. The speaker stated that automatic control of the number of revolutions of a hydrodynamic transformer guarantees that rigidity of the latter will be obtained, which could significantly widen the domain of applicability of hydrodynamic transmission for driving machines and stands, providing a wide range of regulation with approximately constant power.

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G. K. Moskatov

I. Z. Zaichenko (ENIMS, Moscow), in the paper, "On the dynamic stability of pneumatic and pneumo-hydraulic transmissions," spoke of the results of investigations into the dynamic stability of volumetric hydrotransmissions. Comparative data were given for the dynamic characteristics of hydrotransmissions, pneumotransmissions and combined hydropneumatic transmissions.

V. N. Veller (Moscow) presented a paper, "An investigation of various types of piston hydraulic devices." The paper gave several ways of decreasing the adjusting stresses on valves and pistons. An original design of a piston valve was discussed.

L. S. Bron (Moscow) made a report on the subject, "Hydraulic drives for power heads." Results of model research work on the creation of hydraulic systems of self-acting hydraulic power heads for stand assemblies were given.

The paper of V. I. Sherbakov (ENIMS, Moscow), "A new pneumoappliance for driving metal-working stands," was devoted to designs for controlling pneumatic power drives for which automation of the machine's working cycle is guaranteed by just one pneumatic device, without use of electro-automatic elements. Also considered were the building of pneumatic terminal decouplers and pneumovalves.

A. F. Arkhangel'skii, in his paper, "Ways to increase the power of hydraulic universal speed regulators within the previous dimensions," spoke of experiments in increasing the power of oblique-plate hydrostatic transmissions, attained without any increase in physical dimensions, and related the results provided by these experiments.

A. S. Shashkin (Avtomekhanicheskii Institut, Moscow), in reporting on the subject, "Nominal and structural nomenclature of hydrosystem elements," mentioned the difficulties attendant on the study of hydromechanical semi-design schemes, and voiced his hope that standards of nomenclature for hydromechanisms and hydroapparatus would be developed and promulgated in the USSR.

The evening session was given over to a discussion of the papers that had been heard. Taking part in the discussion were E. M. Nadzhafov (IAT, Moscow), V. S. Prusenko (Moscow), P. M. Shanturin (Central Laboratory for Automation, Moscow), Iu. I. Ivlichev (IAT, Moscow), V. Britall (GDR), Io. I. Nikolaev, S. A. Babushkin, V. A. Khokhlov (IAT, Moscow), L. S. Bron (Moscow), B. F. Stupak (The Leningrad Institute of "Sydostroitel'noi Promyshlennosti), Dzh. Kveton (Czechoslovakian National Republic) I. N. Kichin (IAT, Moscow), Lu. Iuan'-tsin (Chinese National Republic), S. M. Zasedatelev (MVTU, Moscow), S. A. Iushchenko (The "Manometr" Factory, Moscow) and Ian Khampl (Czechoslovakian National Republic).

M. A. Aizerman summed up the work of the Second Meeting of the All-Union Seminar on Pneumo-Hydraulic Automata. He noted that the Proceedings of the First Meeting would be out in the near future, and spoke of the preparations for publishing the proceedings of the present meeting.

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